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A Piecewise Cubic PostScript Trefoil — F.E.J. Linton

Math/CS Emeritus, Wesleyan Univ., Middletown, CT, USA

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Trefoil Parameterizations

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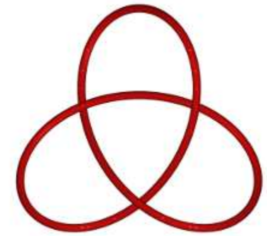

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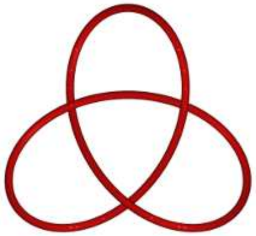

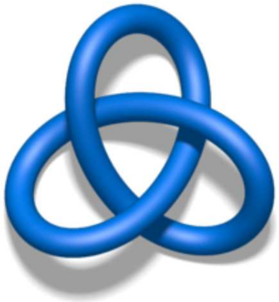

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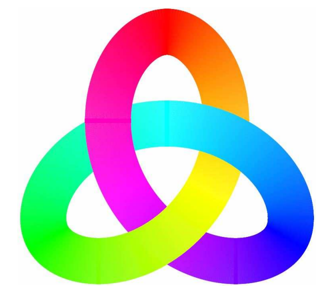

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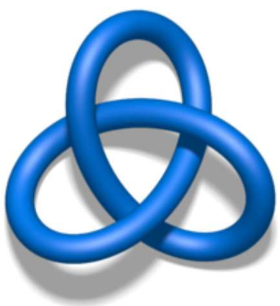

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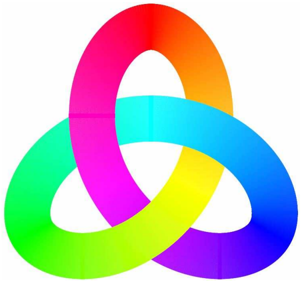

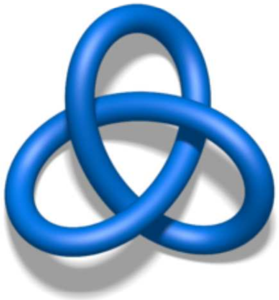

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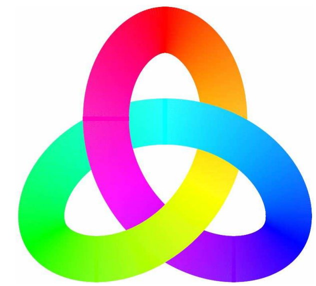

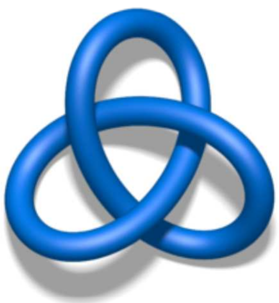

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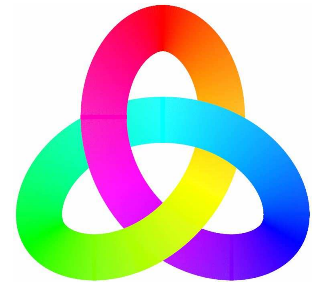

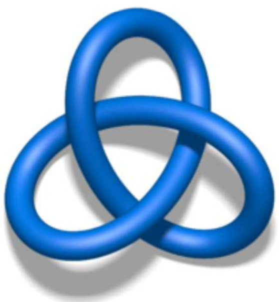

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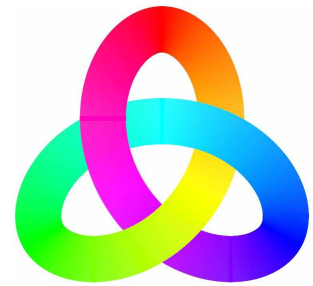

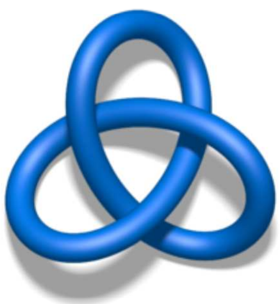

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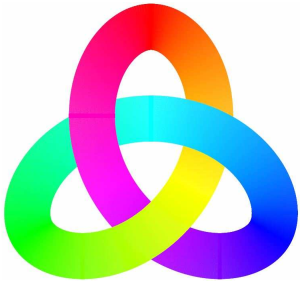

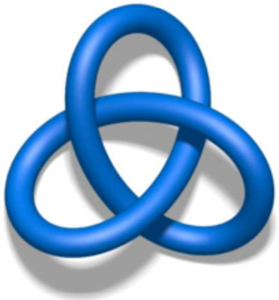

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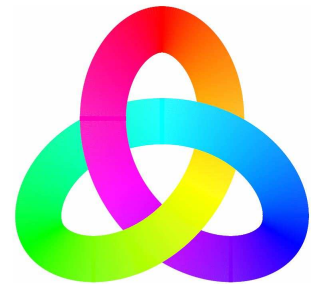

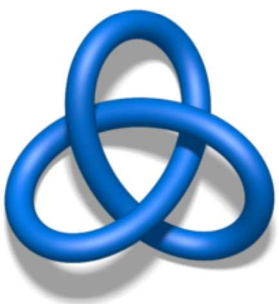

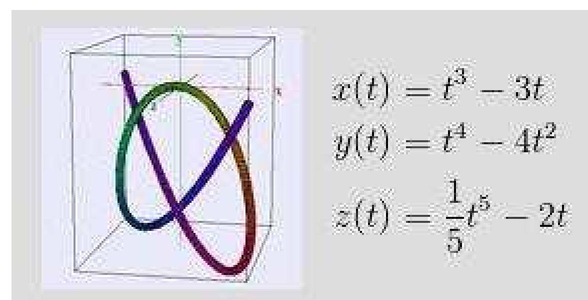

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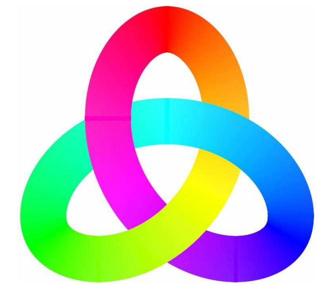

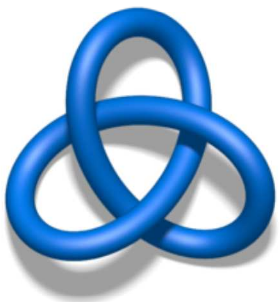

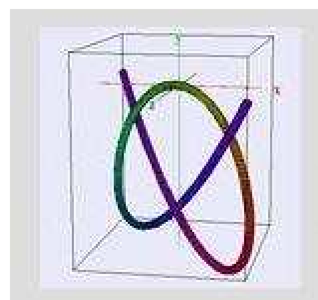

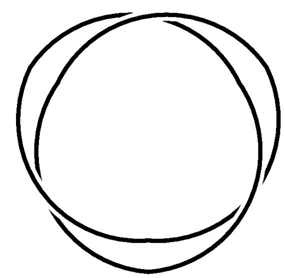

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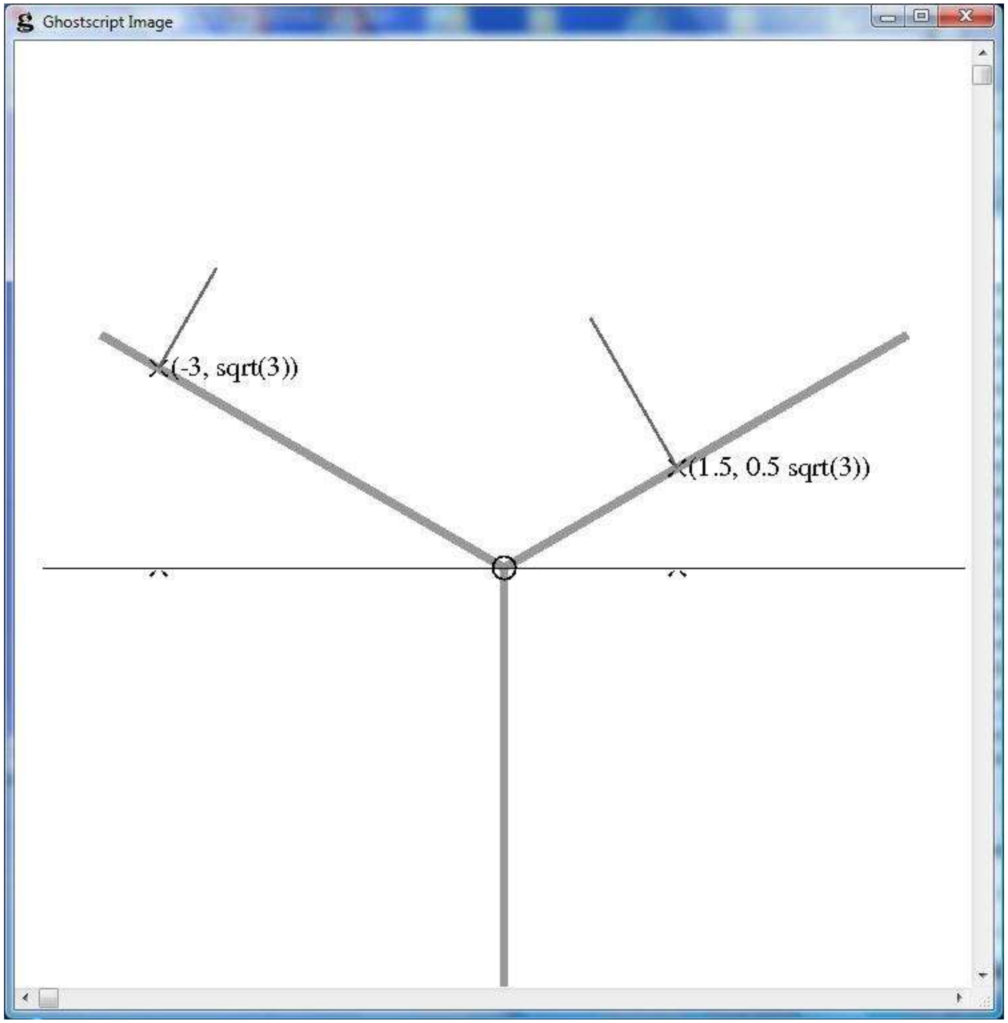


Begin with a fragment of the graph of

$y=|x|/\sqrt{3}$,

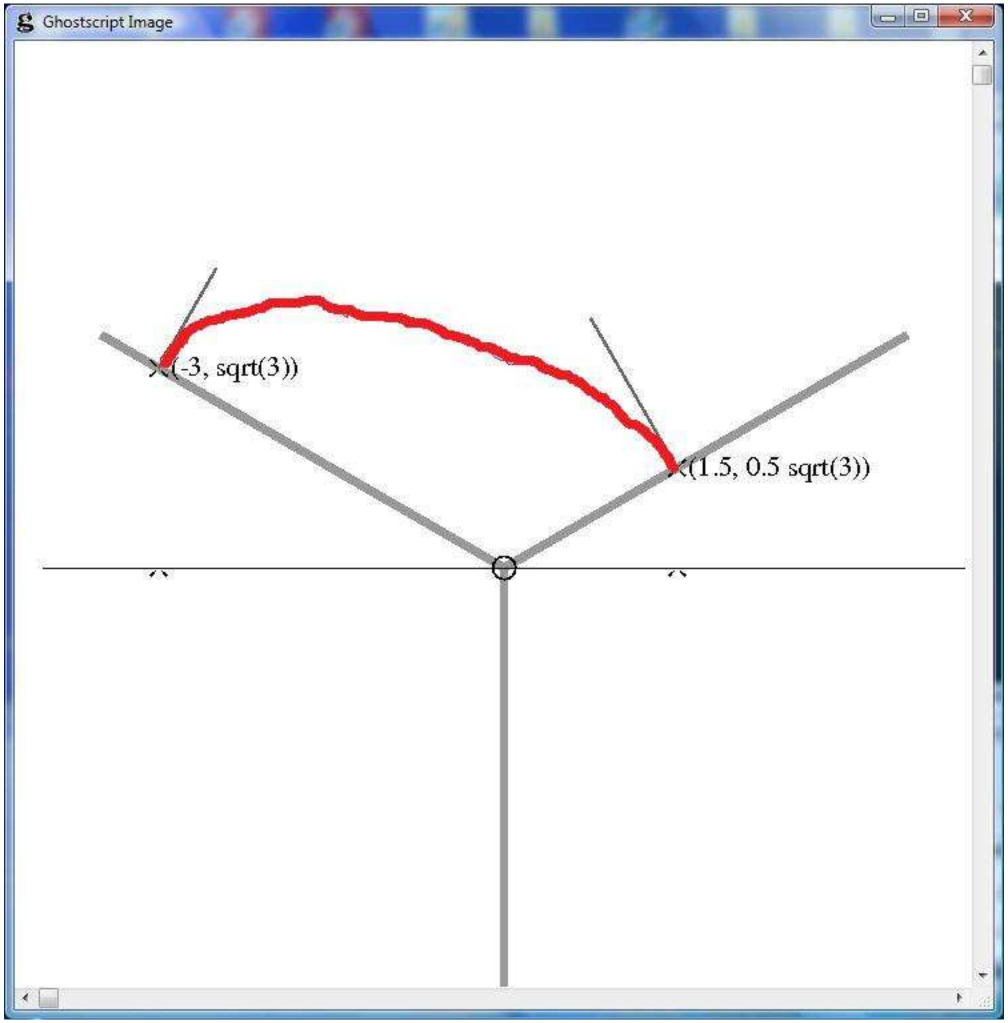
with origin, a vertical y-axis, and perpendiculars at

$(x_0, y_0)=(-3, \sqrt{3})$

and

$(x_3, y_3)=(1.5, \frac{1}{2}\sqrt{3})$,

all as marked.



Sketched in red, a part of the graph of a function `y = p(x)` satisfying

`p(-3)= √3,`

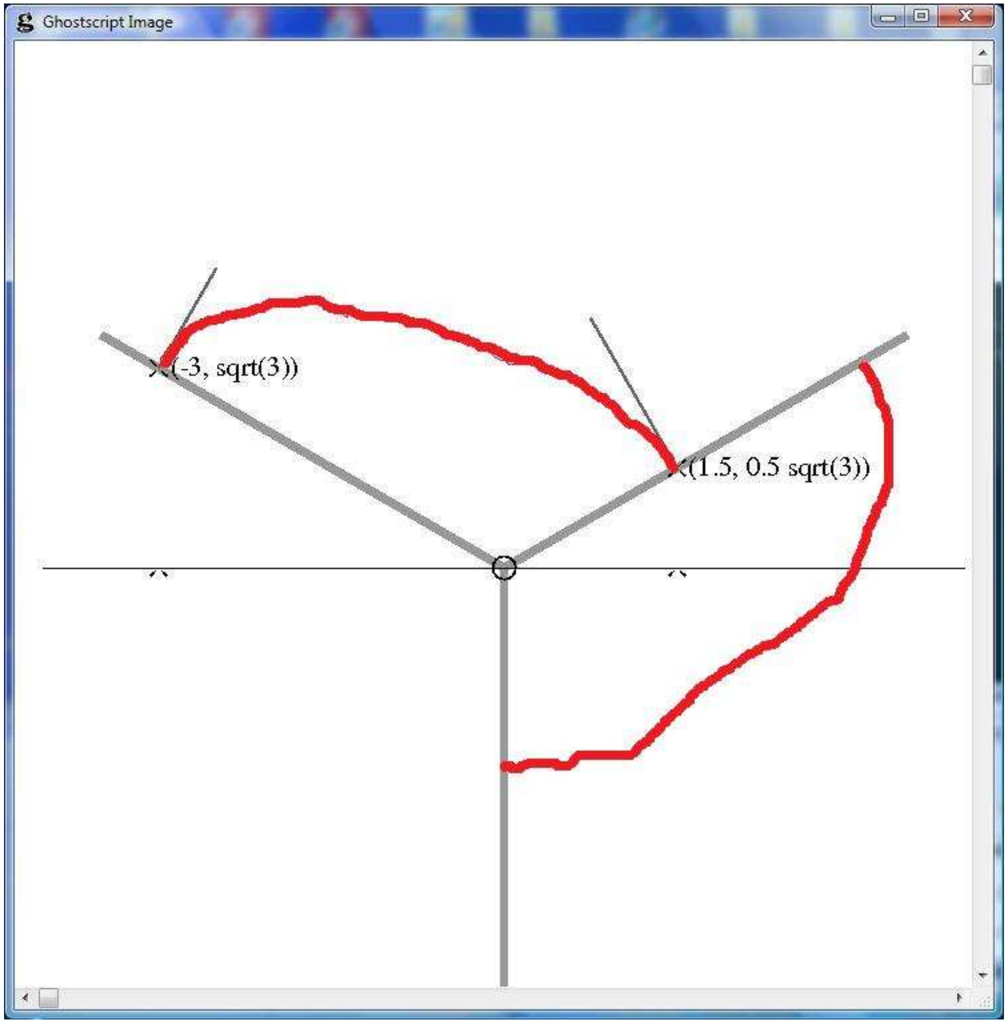
`p'(-3)= √3,`

`p(1.5)= ½√3,`

and

`p'(1.5)= -√3.`

For ...



For if we rotate the sketched arc not just once through 2π/3 radians, ...

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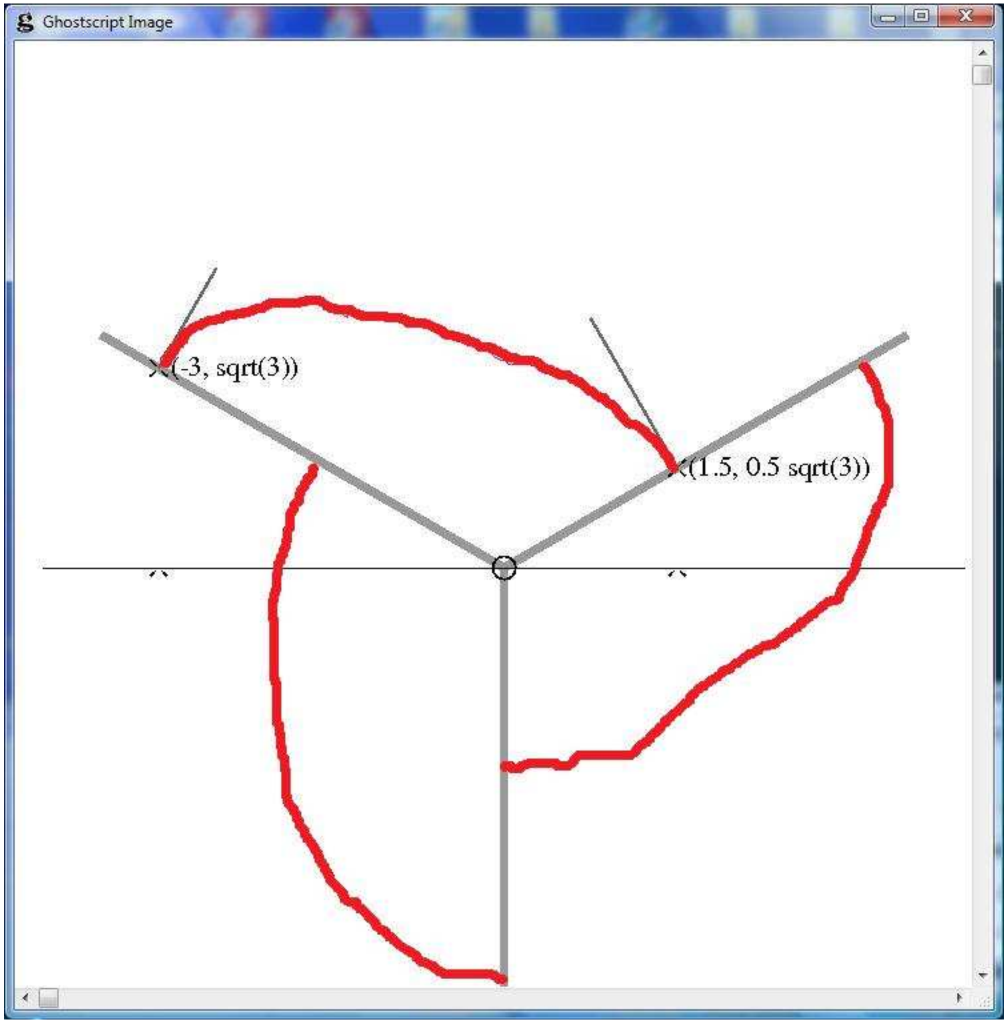
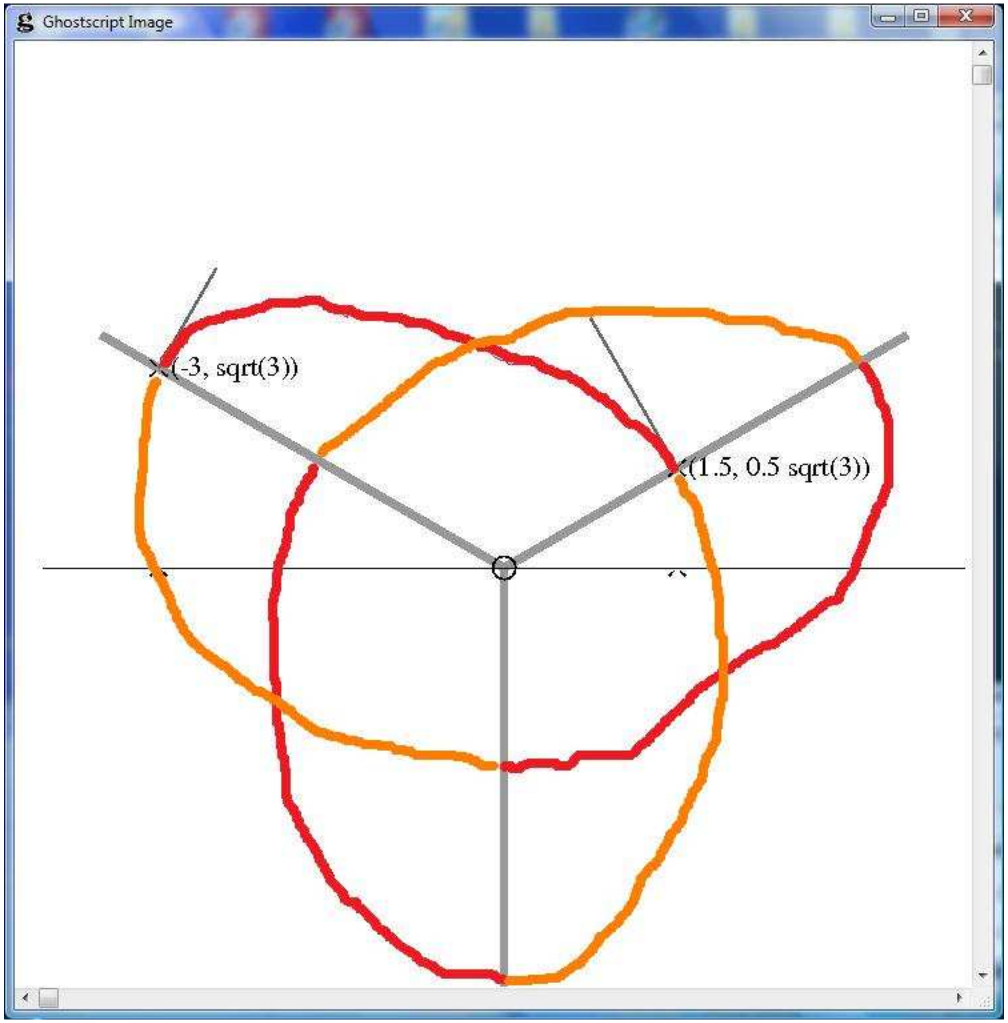
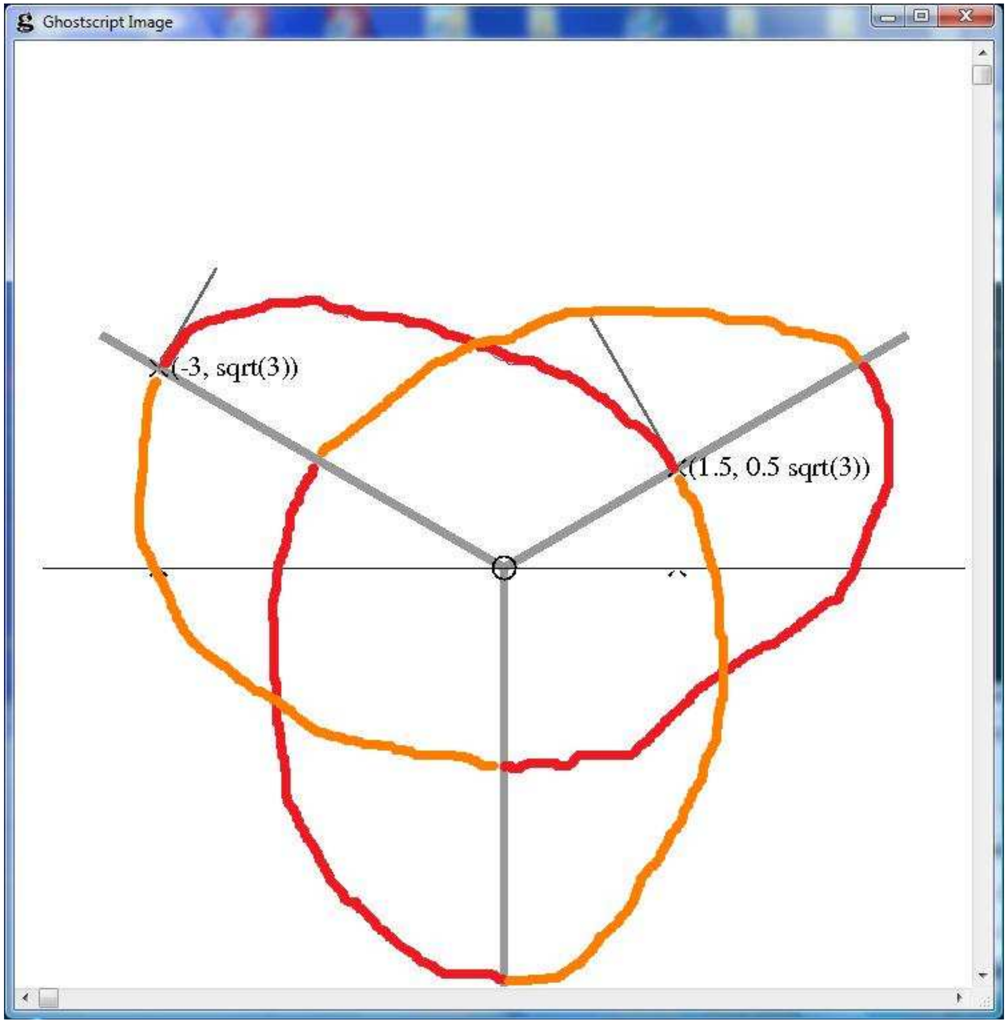


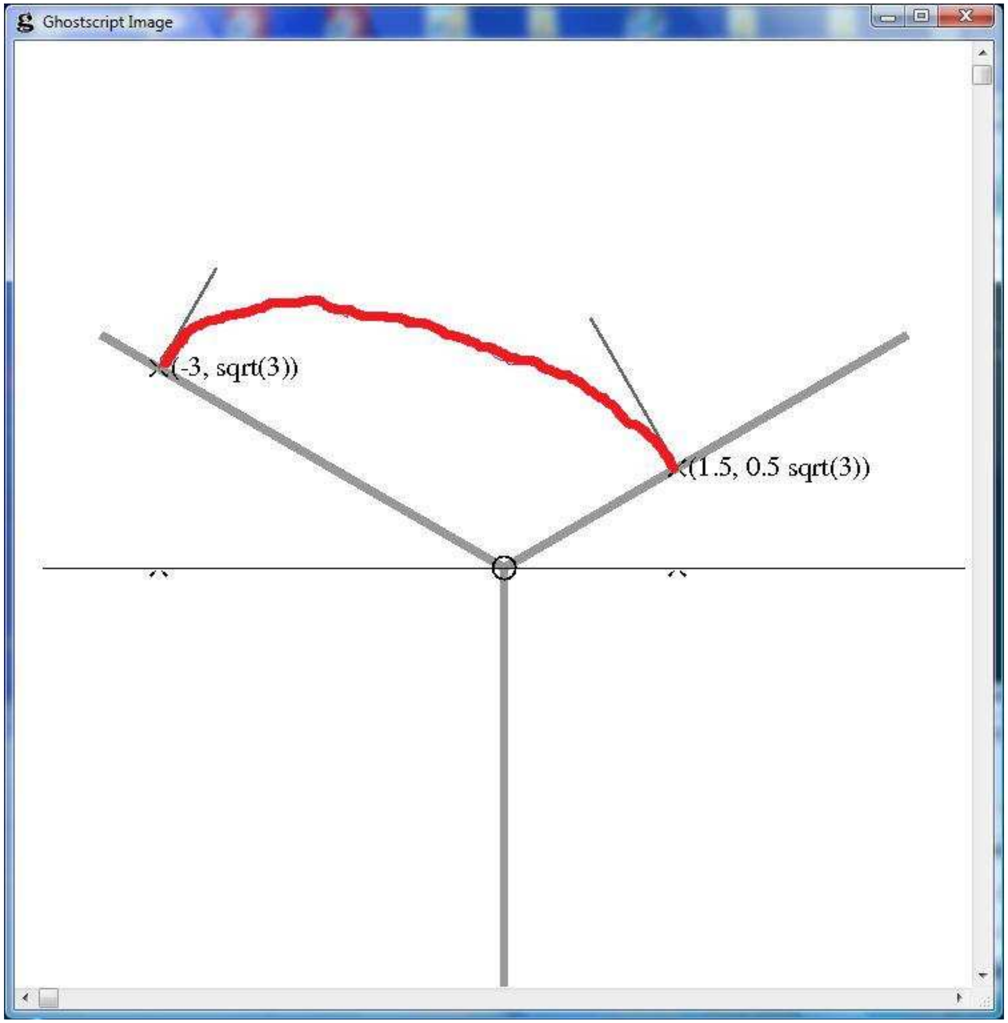
For if we rotate the sketched arc not just once through 2π/3 radians, but twice, and then ...



For if we rotate the sketched arc not just once through 2π/3 radians, but twice, and then superimpose the mirror image of that result, ...



For if we rotate the sketched arc not just once through 2π/3 radians, but twice, and then superimpose the mirror image of that result, we'd have the sort of trefoil we're after.



Now to find coefficients `a, b, c, d` for a cubic polynomial

`p(x) = ax`^{3}`+bx`^{2}`+cx+d` ,

satisfying

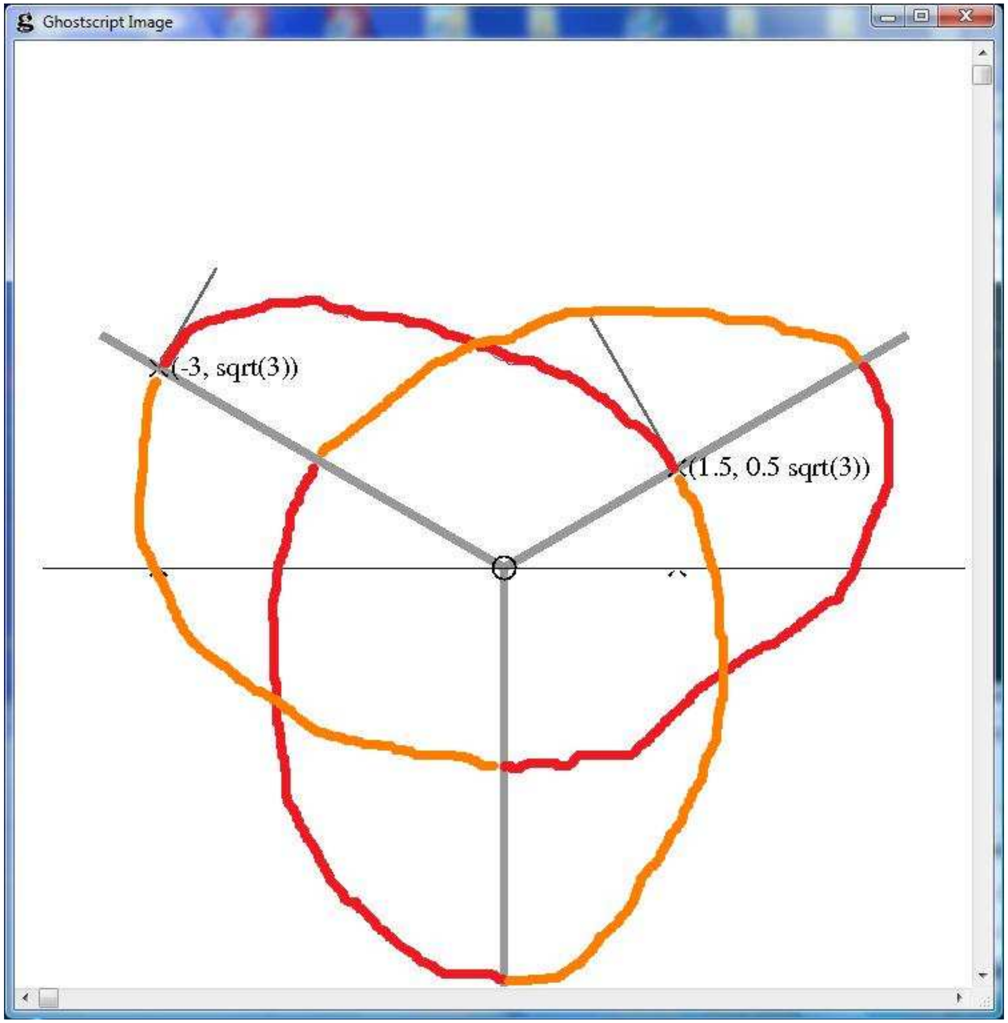
`p(-3) = √3,`

`p'(-3) = √3,`

`p(1.5) = ½√3,` and

`p'(1.5) = -√3`

is standard Linear Algebra, as is ...



... rotating and reflecting its graph in the plane.

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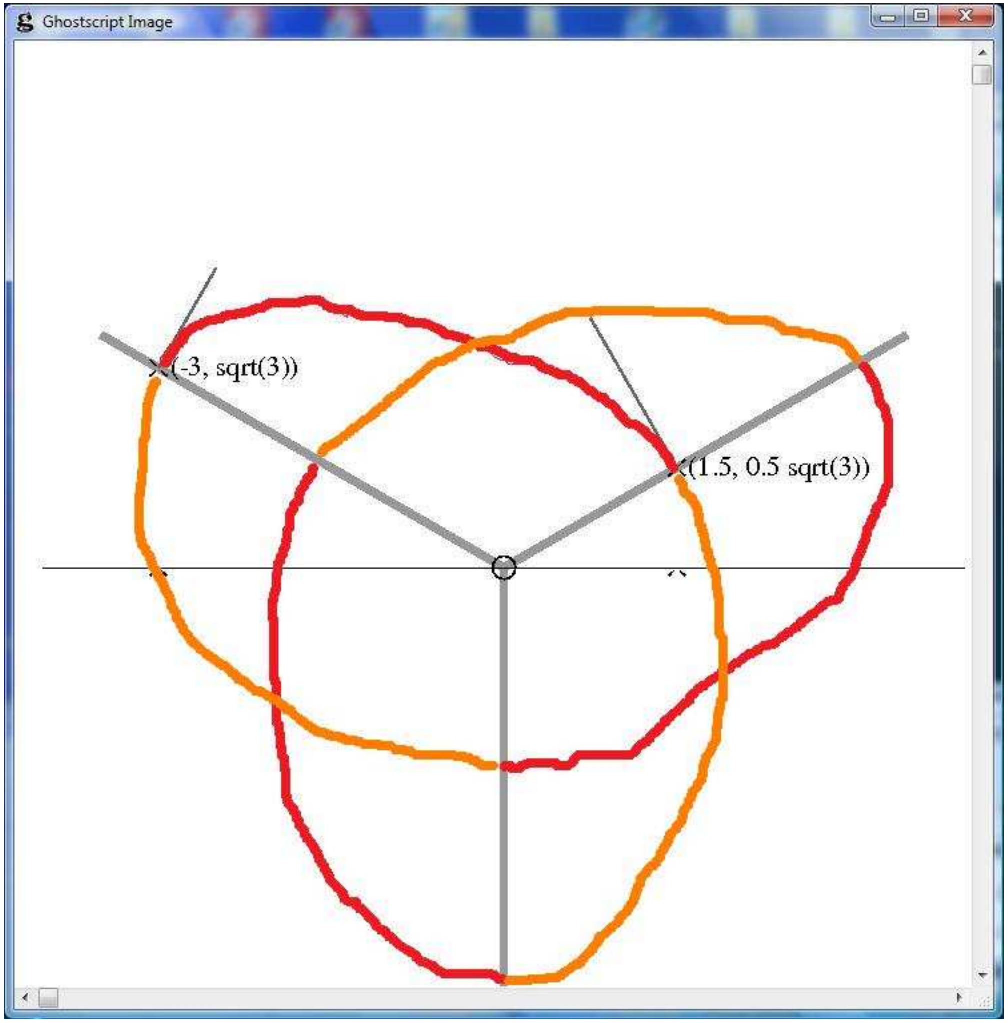

... rotating and reflecting its graph in the plane. Fortunately, PostScript language can spare you many matrix calculations. For here's how PostScript draws cubic polynomials (crash course part 1):

Mathematically, a cubic Bézier curve is derived from a pair of parametric cubic equations:

$$x(t) = a_x t^3 + b_x t^2 + c_x t + x_0$$

$$y(t) = a_y t^3 + b_y t^2 + c_y t + y_0$$

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In our case, we have $a_x=0=b_x$, $c_x=4.5$, $x_0=-3$, $y(t)=p(x(t))$, and $y_0=\sqrt{3}$.

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In our case, we have $a_x=0=b_x$, $c_x=4.5$, $x_0=-3$, $y(t)=p(x(t))$, and $y_0=\sqrt{3}$.
The PostScript Bézier curve drawing process uses $x_n=x(nt/3)$ and $y_n=y(nt/3)$, for $n=0,1,2,3$. The middle two (with $n=1,2$) are the so-called *control* points.

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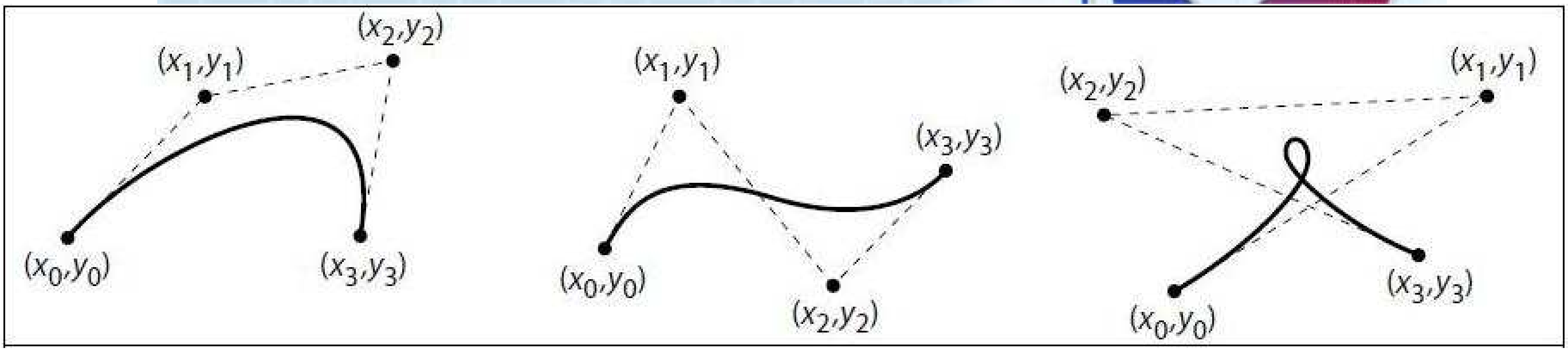
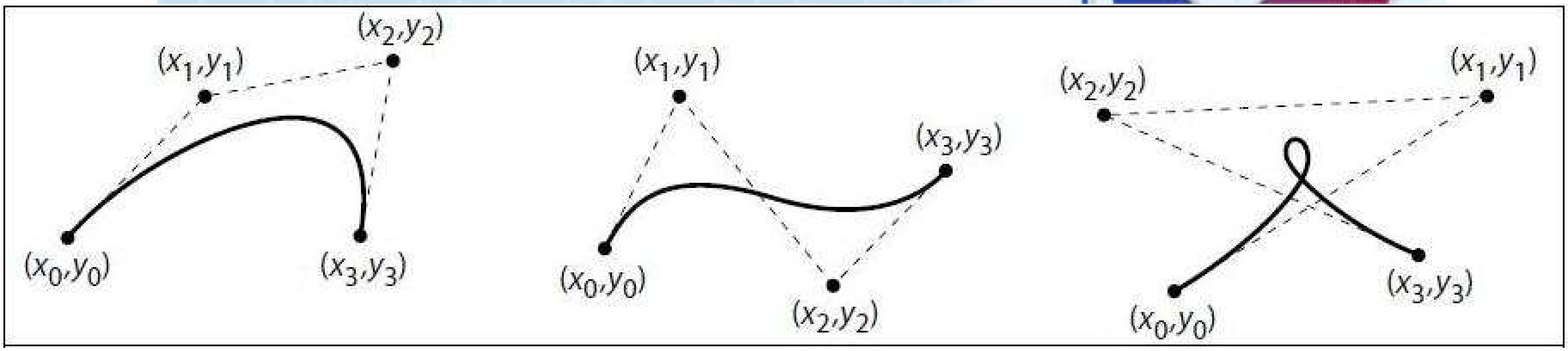


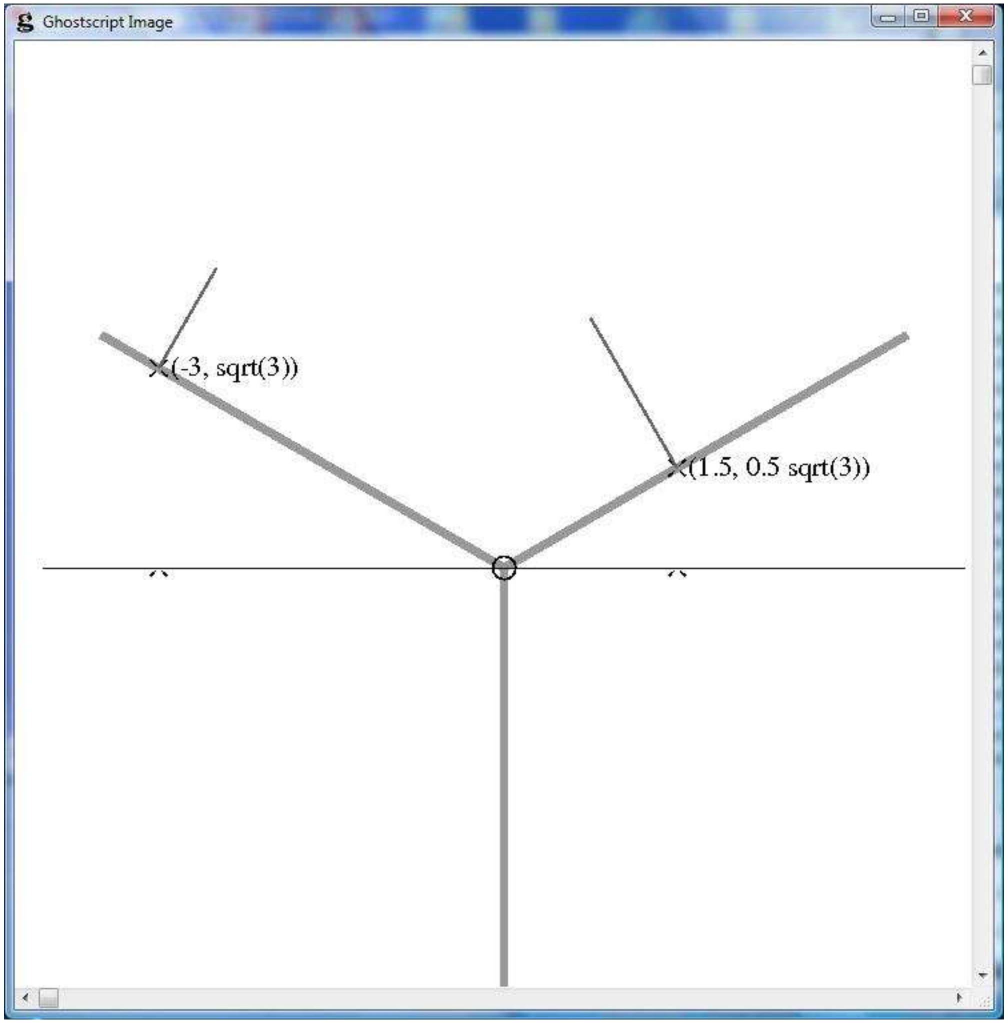
In our case, we have $a_x=0=b_x$, $c_x=4.5$, $x_0=-3$, $y(t)=p(x(t))$, and $y_0=\sqrt{3}$.
The PostScript Bézier curve drawing process uses $x_n=x(nt/3)$ and $y_n=y(nt/3)$, for $n=0,1,2,3$. The curve departs from (x_0, y_0) in the direction of (x_1, y_1) , and arrives at (x_3, y_3) from the direction of (x_2, y_2) .



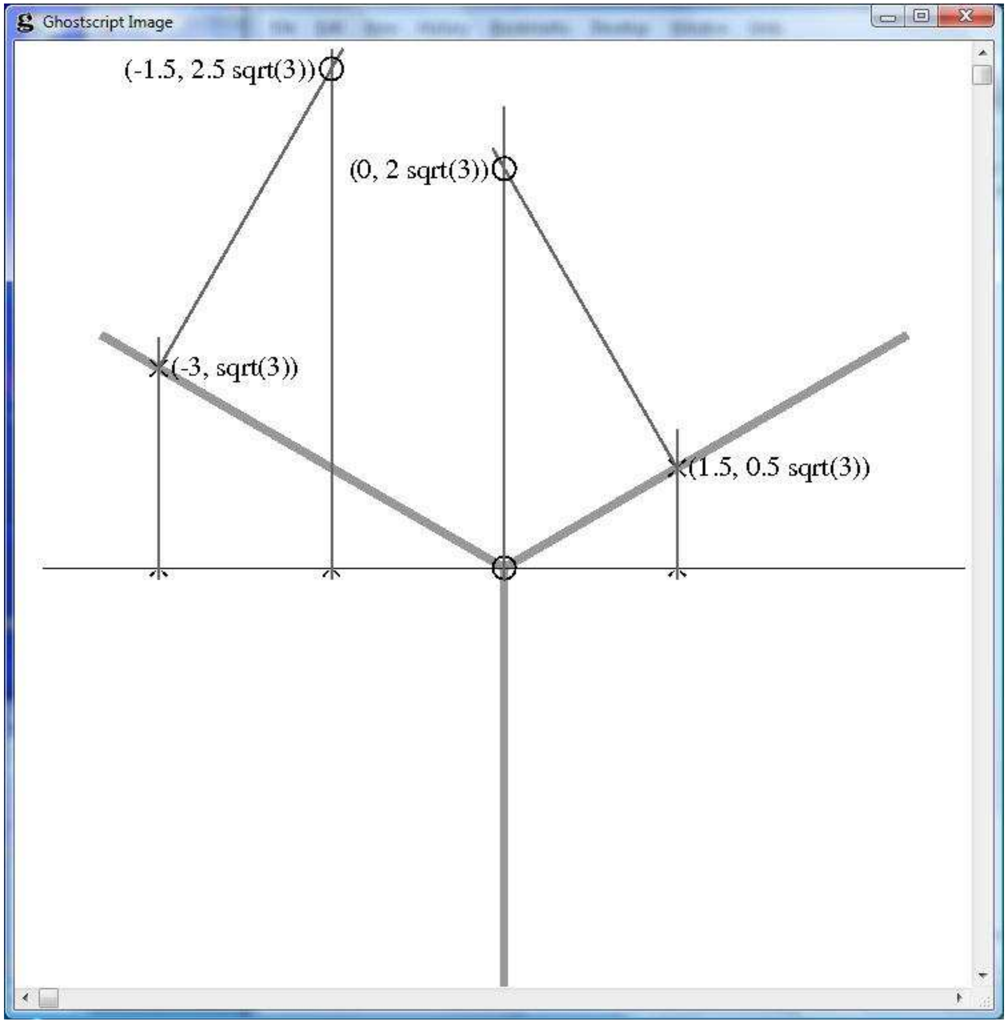
In our case, we have $a_x = 0 = b_x$, $c_x = 4.5$, $x_0 = -3$, $y(t) = p(x(t))$, and $y_0 = \sqrt{3}$.

The PostScript Bézier curve drawing process uses $x_n = x(nt/3)$ and $y_n = y(nt/3)$, for $n = 0,1,2,3$. The curve departs from (x_0, y_0) in the direction of (x_1, y_1), and arrives at (x_3, y_3) from the direction of (x_2, y_2).

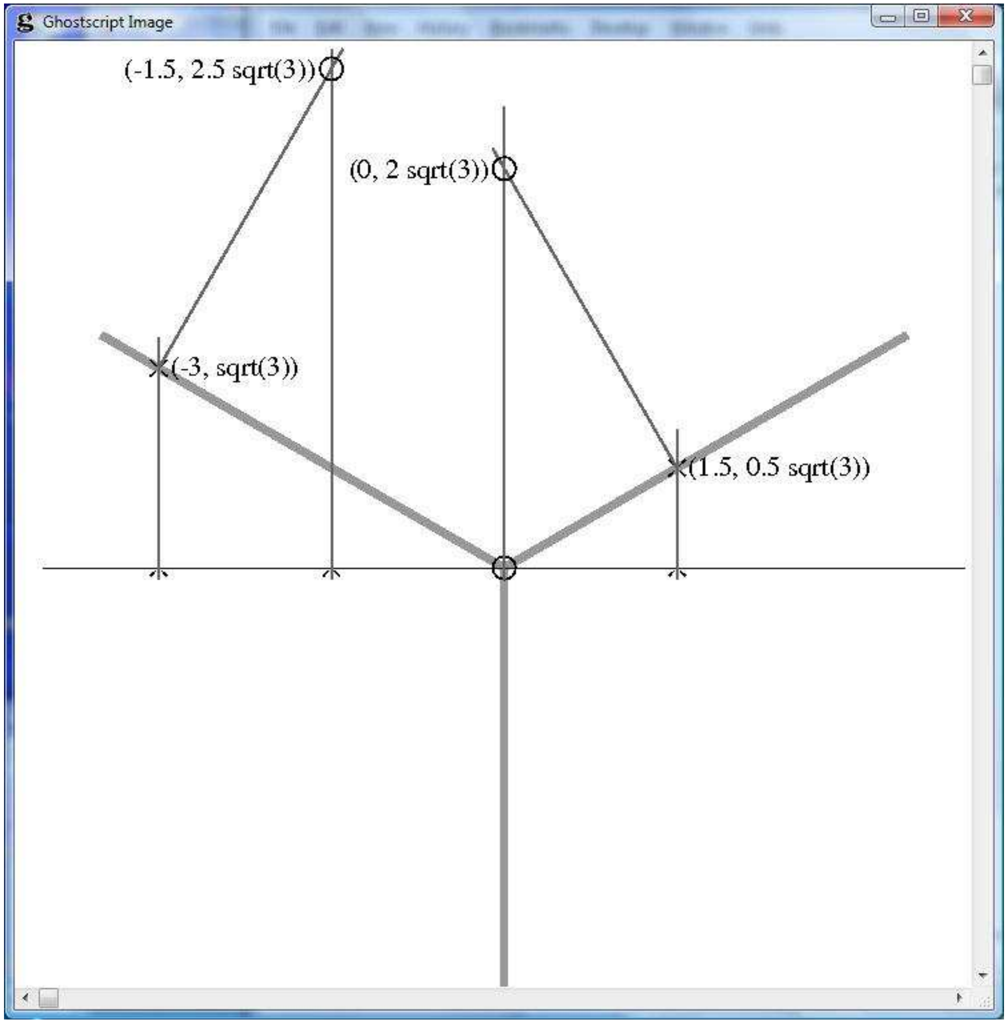
The “control points” we need are easy to construct:



We already have the graph of $y=|x|/\sqrt{3}$, the origin, the start and end pts $(x_0, y_0)=(-3, \sqrt{3})$, $(x_3, y_3)=(1.5, \frac{1}{2}\sqrt{3})$, a vertical y-axis, and orthogonals at $(x_0, y_0)=(-3, \sqrt{3})$ and $(x_3, y_3)=(1.5, \frac{1}{2}\sqrt{3})$, as marked.



We construct the auxiliary line segments that will meet at the control points over $x_1=-1.5$ and $x_2=0$, the control points becoming $(x_1, y_1)=(-1.5, 2.5\sqrt{3})$ and $(x_2, y_2)=(0, 2\sqrt{3})$, respectively.



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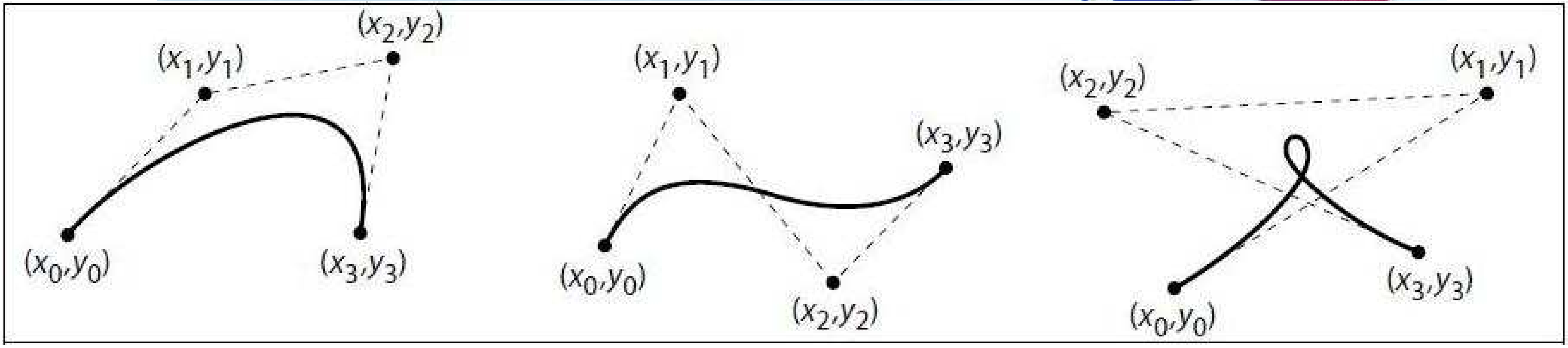
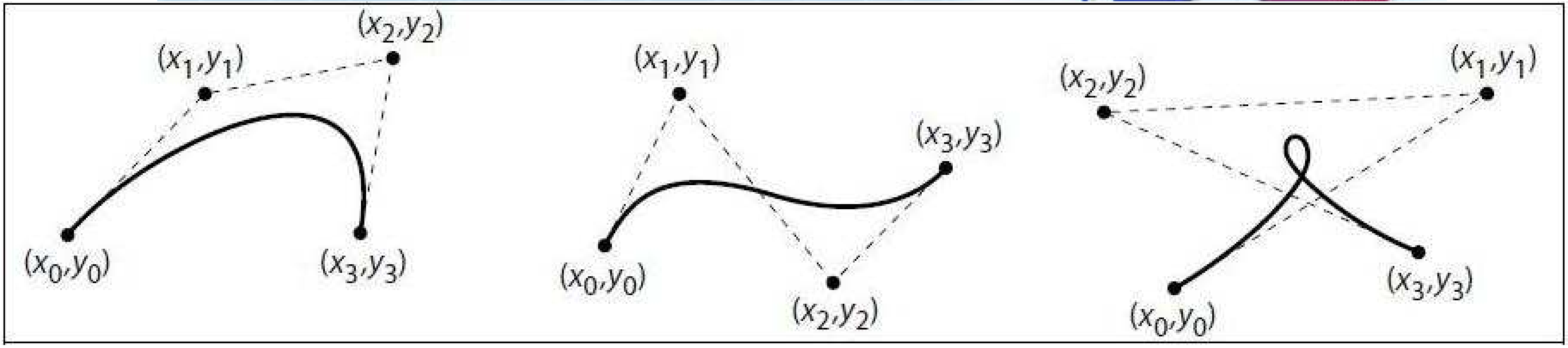


If PostScript's drawing finger is pointed at `(`x_0`, `y_0`)` , here's how to draw the Bézier curve to `(`x_3`, `y_3`)` using control points `(`x_1`, `y_1`)` and `(`x_2`, `y_2`)` —



If PostScript's drawing finger is pointed at (x_0, y_0) , here's how to draw the Bézier curve to (x_3, y_3) using control points (x_1, y_1) and (x_2, y_2) — use **`curveto`** :

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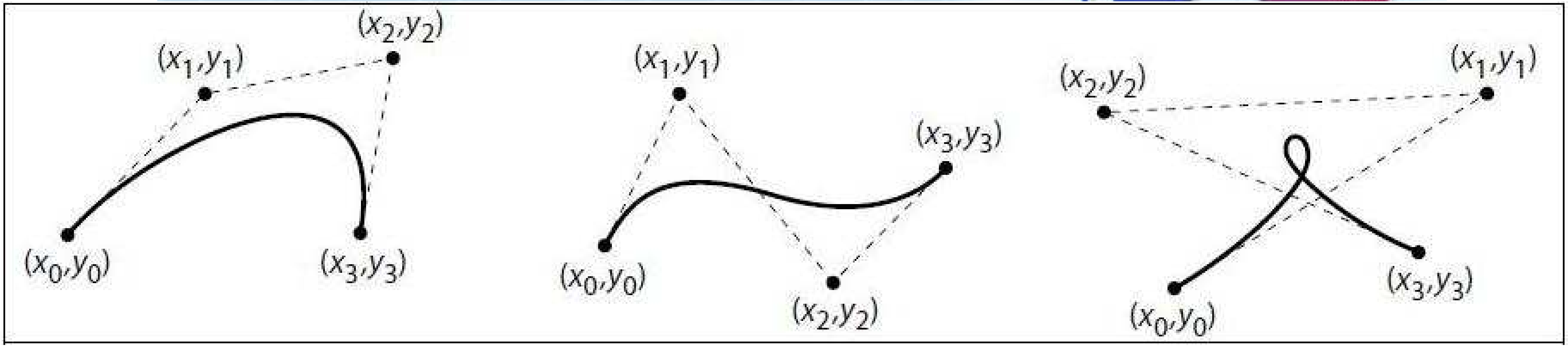
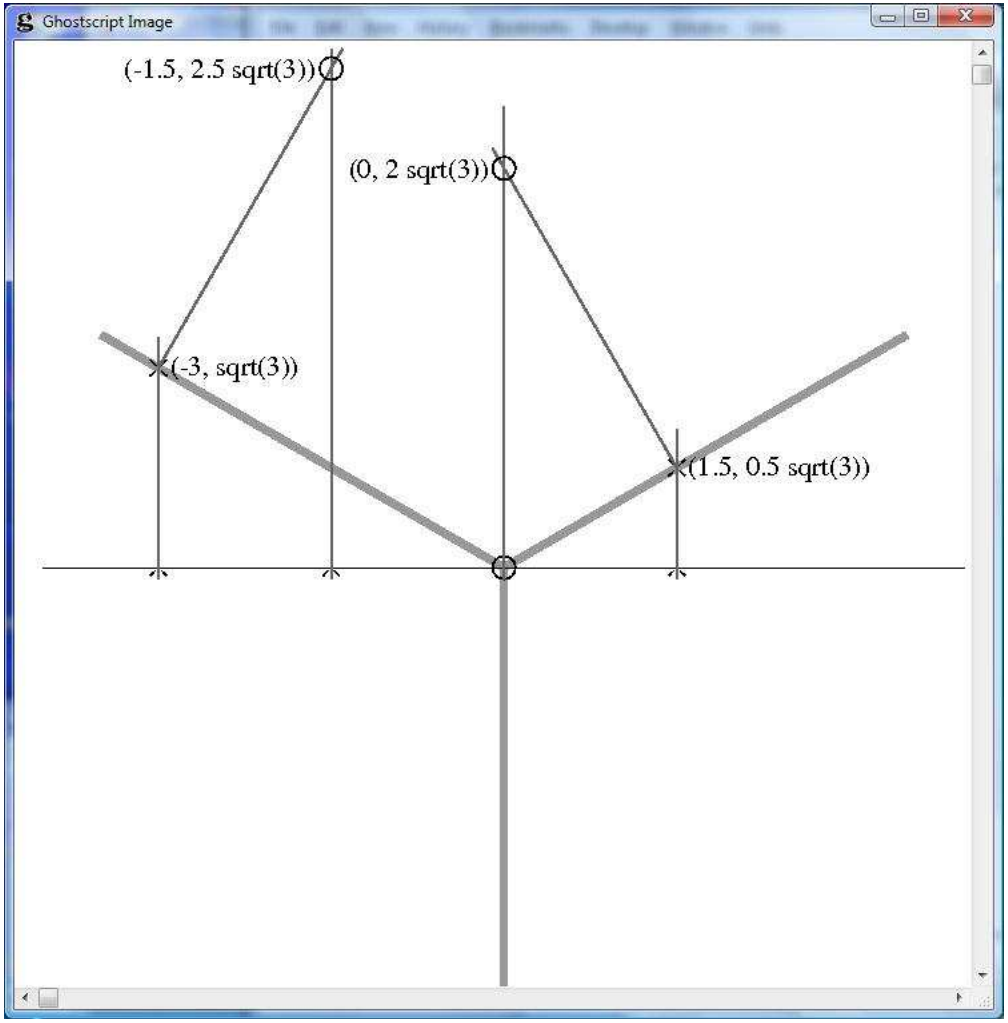


If PostScript's drawing finger is pointed at (x_0, y_0) , here's how to draw the Bézier curve to (x_3, y_3) using control points (x_1, y_1) and (x_2, y_2) — use **`curveto`** :

curveto $x_1\ y_1\ x_2\ y_2\ x_3\ y_3$ **curveto** –

appends a section of a cubic Bézier curve to the current path between the current point (x_0, y_0) and the endpoint (x_3, y_3), using (x_1, y_1) and (x_2, y_2) as the Bézier control points. The endpoint (x_3, y_3) becomes the new current point.



So we must move PostScript's writing finger to `(-3, √3)` and then perform the RPN-based PS incantation

```
-1.5 2.5 3 sqrt mul
0 2 3 sqrt mul
1.5 0.5 3 sqrt mul
curveto
```

The result:

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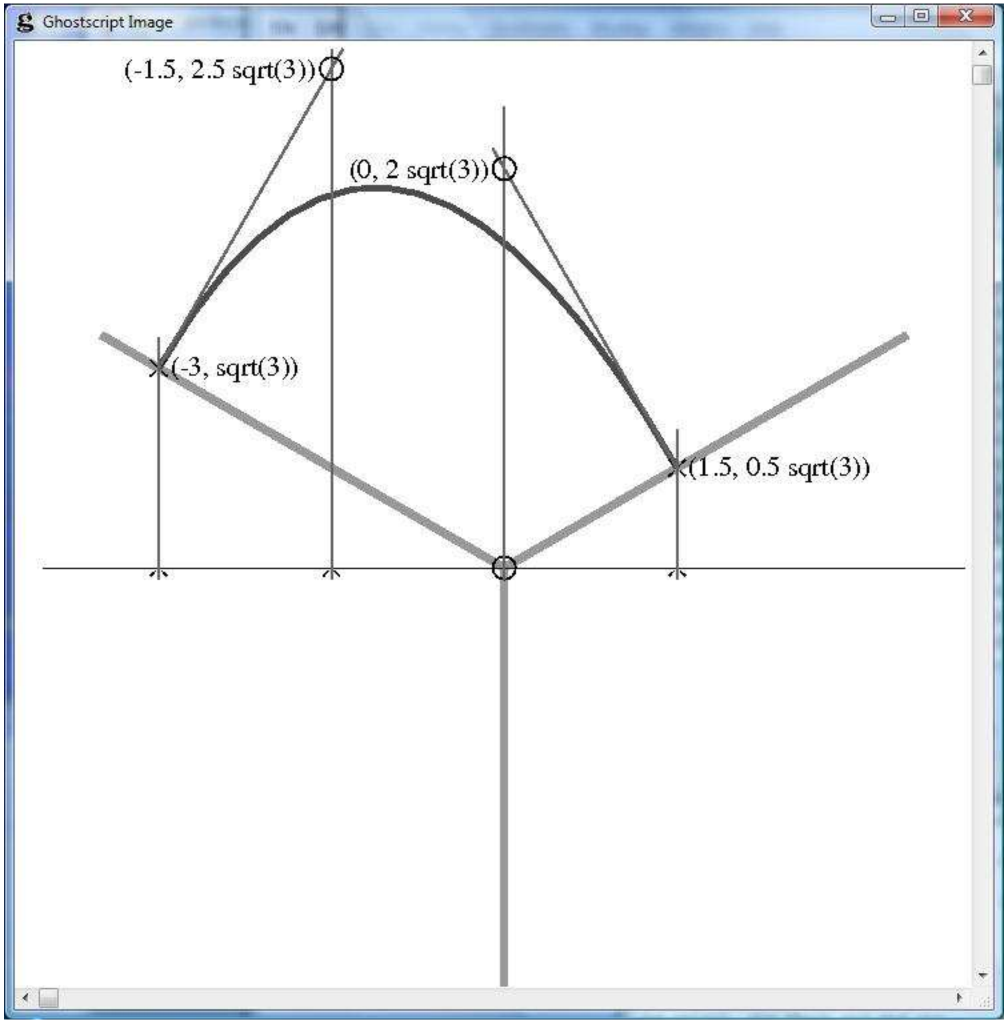

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```
-1.5 2.5 3 sqrt mul
0 2 3 sqrt mul
1.5 0.5 3 sqrt mul
curveto
```

The result: (that was easy).

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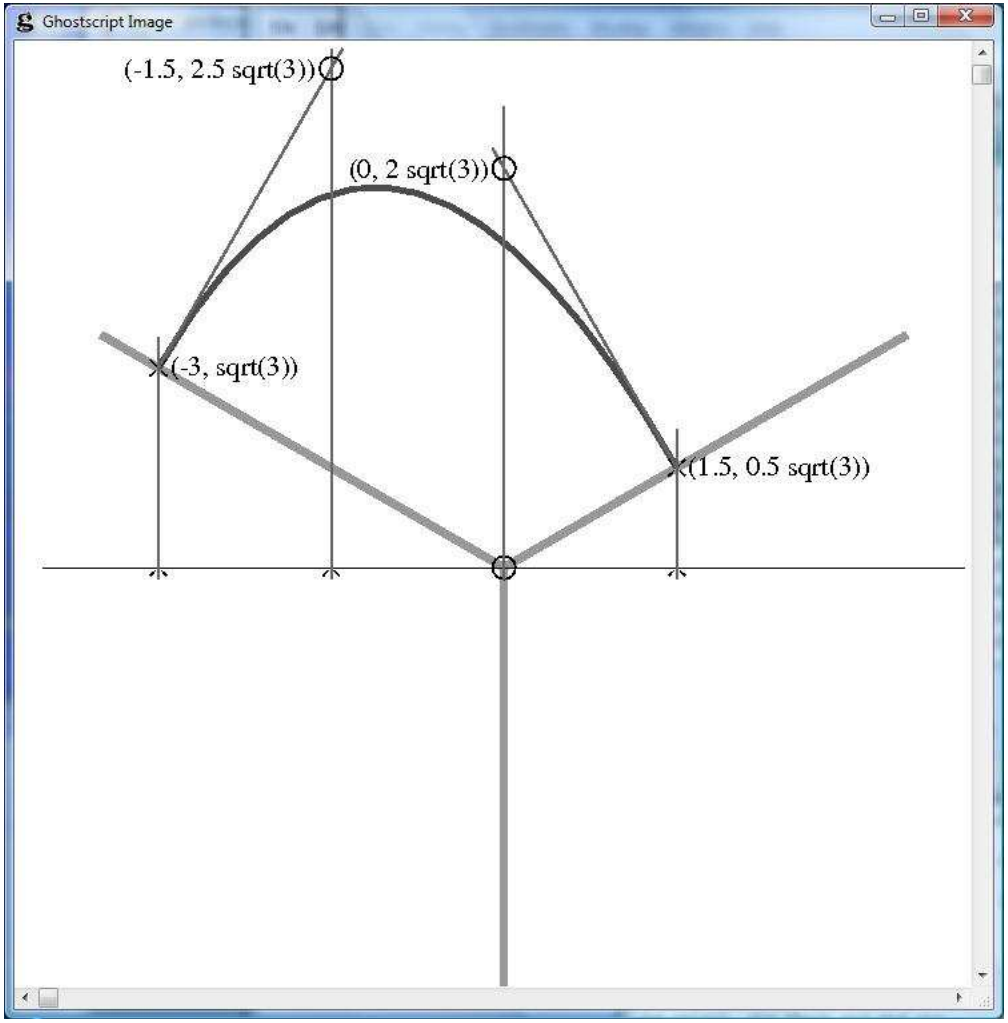

We do the same for the analogous points between `0.6·(-3, √3)` and `0.1·(1.5, ½√3)`.

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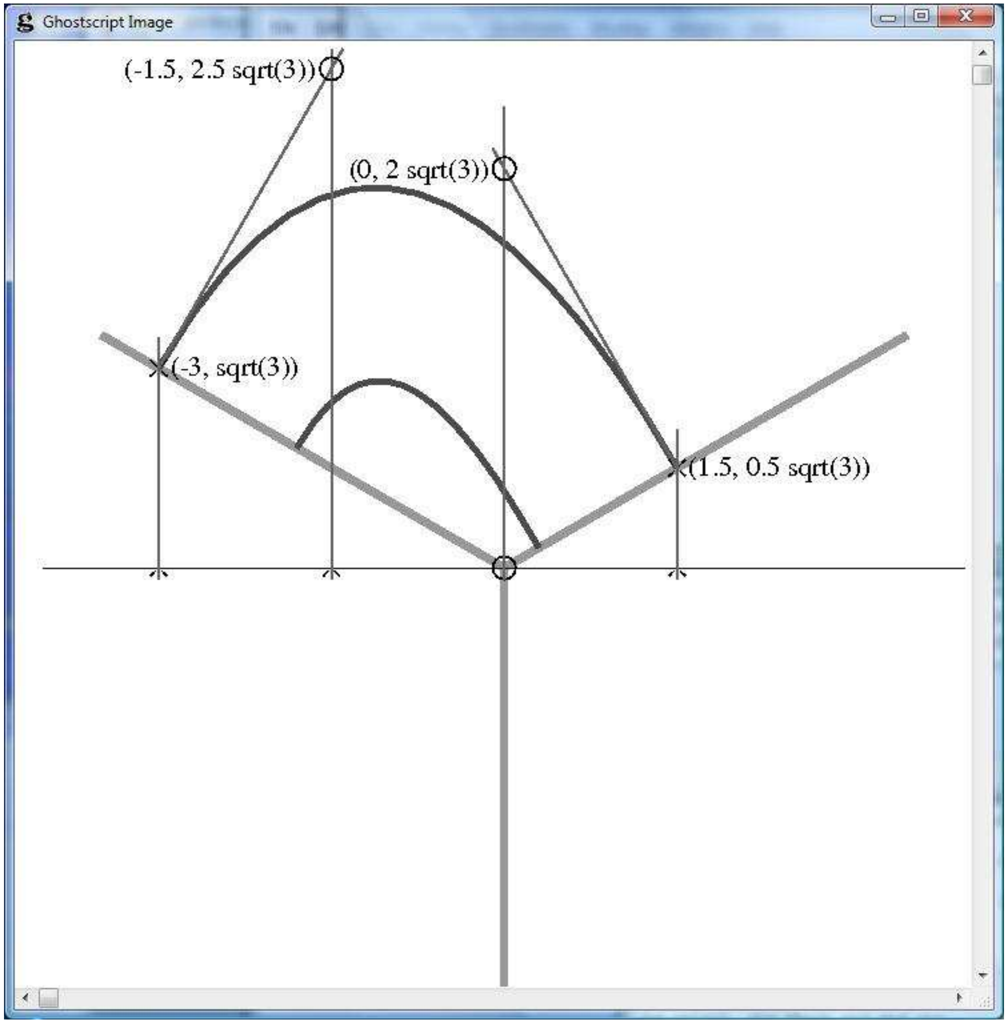

We do the same for the analogous points between `0.6·(-3, √3)` and `0.1·(1.5, ½√3)` .

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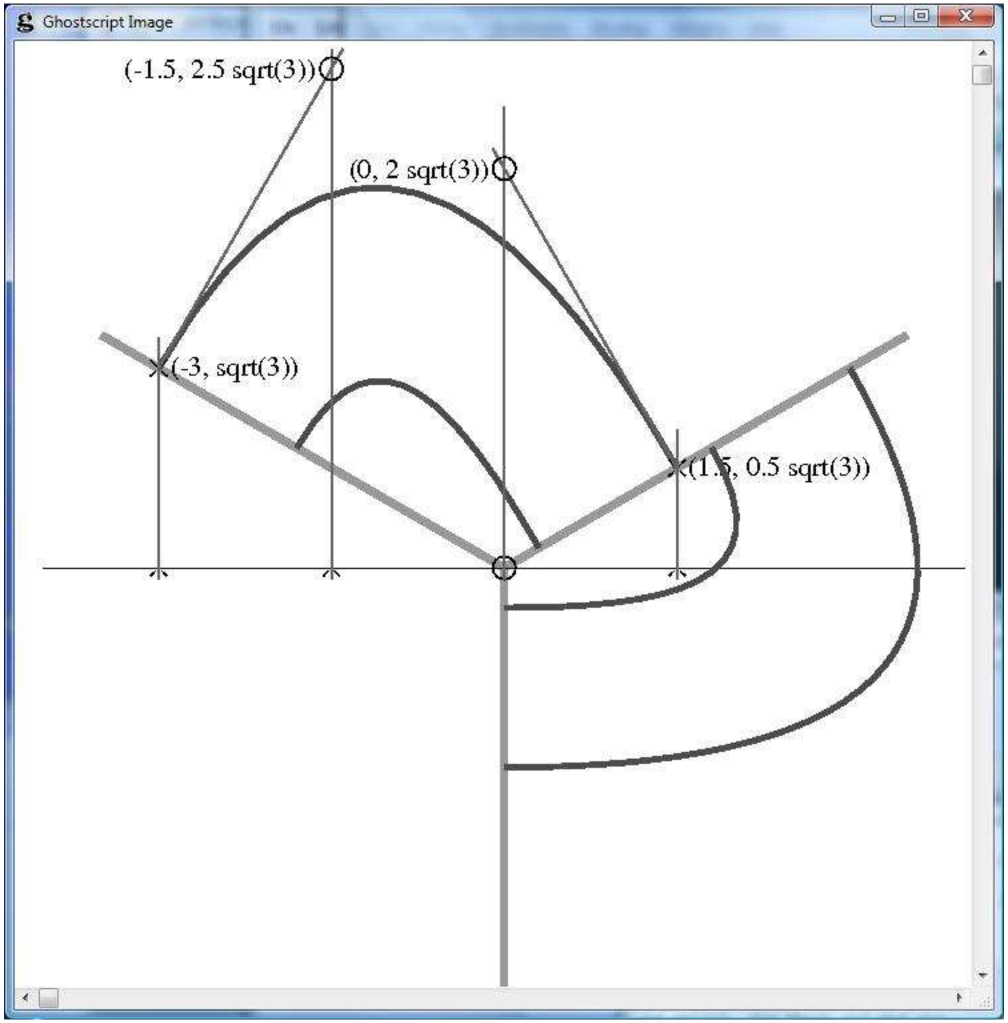

We do the same for the analogous points between `0.6·(-3, √3)` and `0.1·(1.5, ½√3)` .
And then we'll rotate ... once ...

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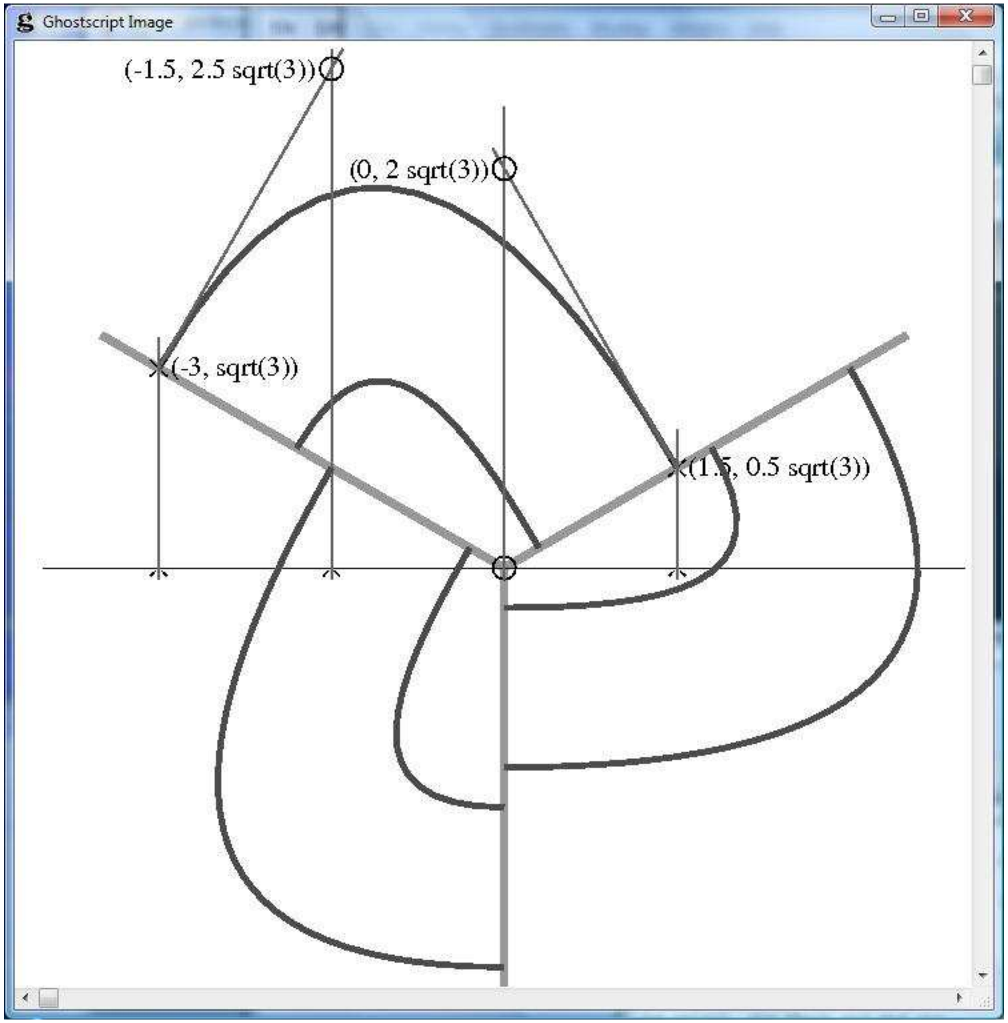

We do the same for the analogous points between `0.6·(-3, √3)` and `0.1·(1.5, ½√3)` . And then we'll rotate ... once ... and twice ...

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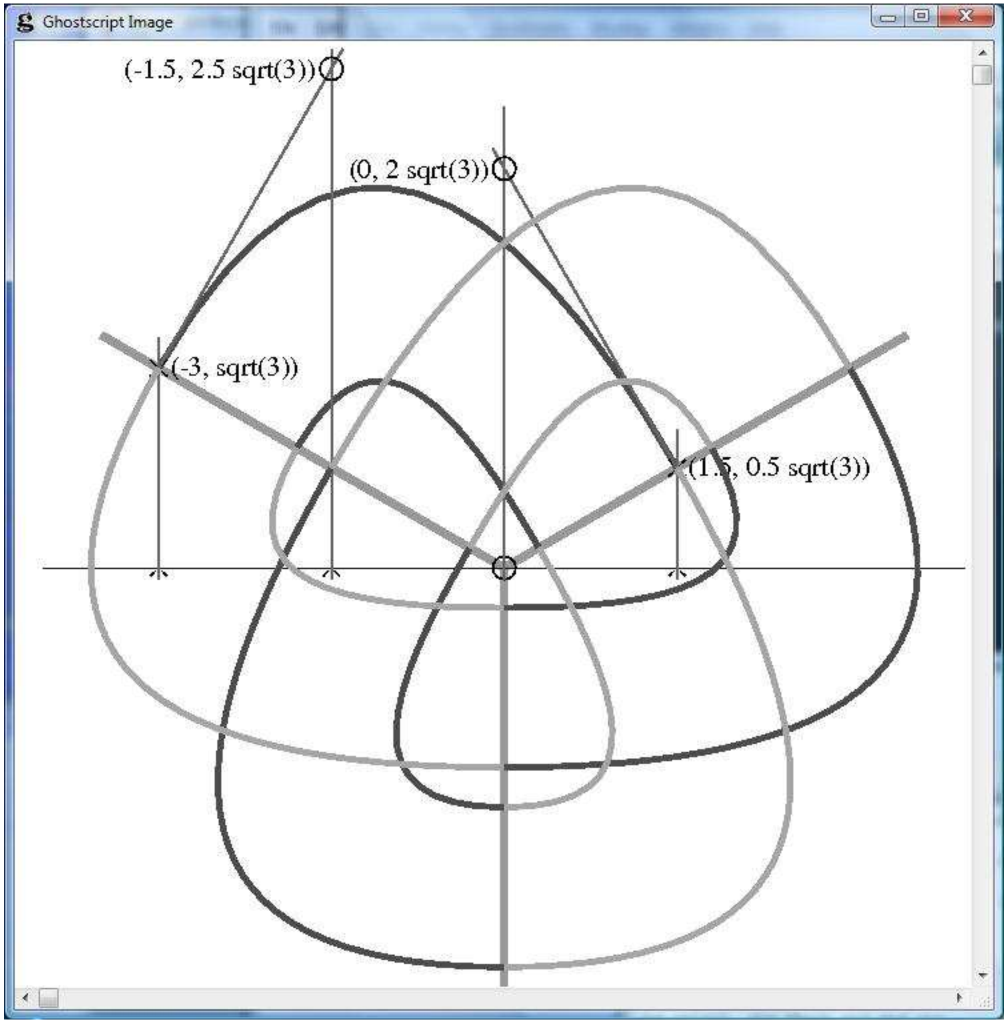

We do the same for the analogous points between `0.6·(-3, √3)` and `0.1·(1.5, ½√3)` . And then we'll rotate ... once ... and twice ... and then reflect across the `y`-axis.

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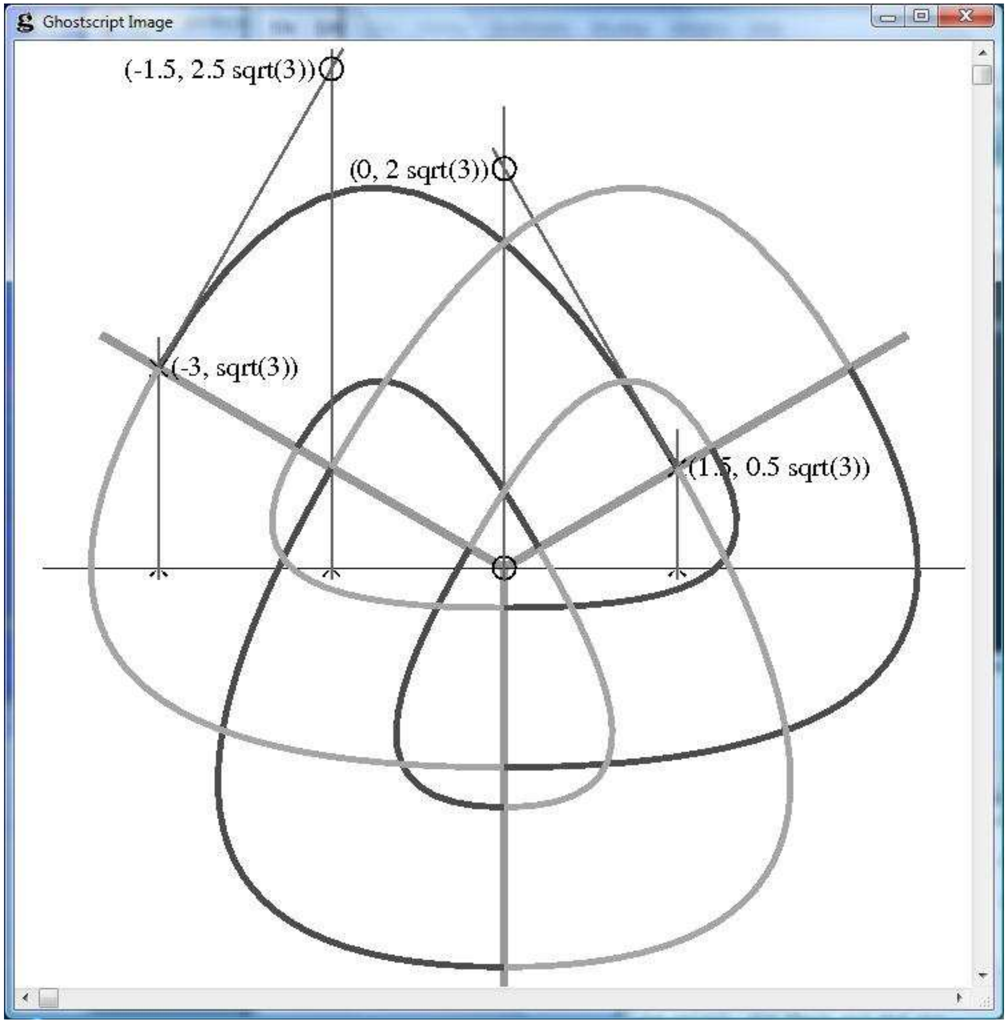

We do the same for the analogous points between $0.6\cdot(-3, \sqrt{3})$ and $0.1\cdot(1.5, \frac{1}{2}\sqrt{3})$. And then we'll rotate ... once ... and twice ... and then reflect across the y-axis. Now let's clean up a little ...

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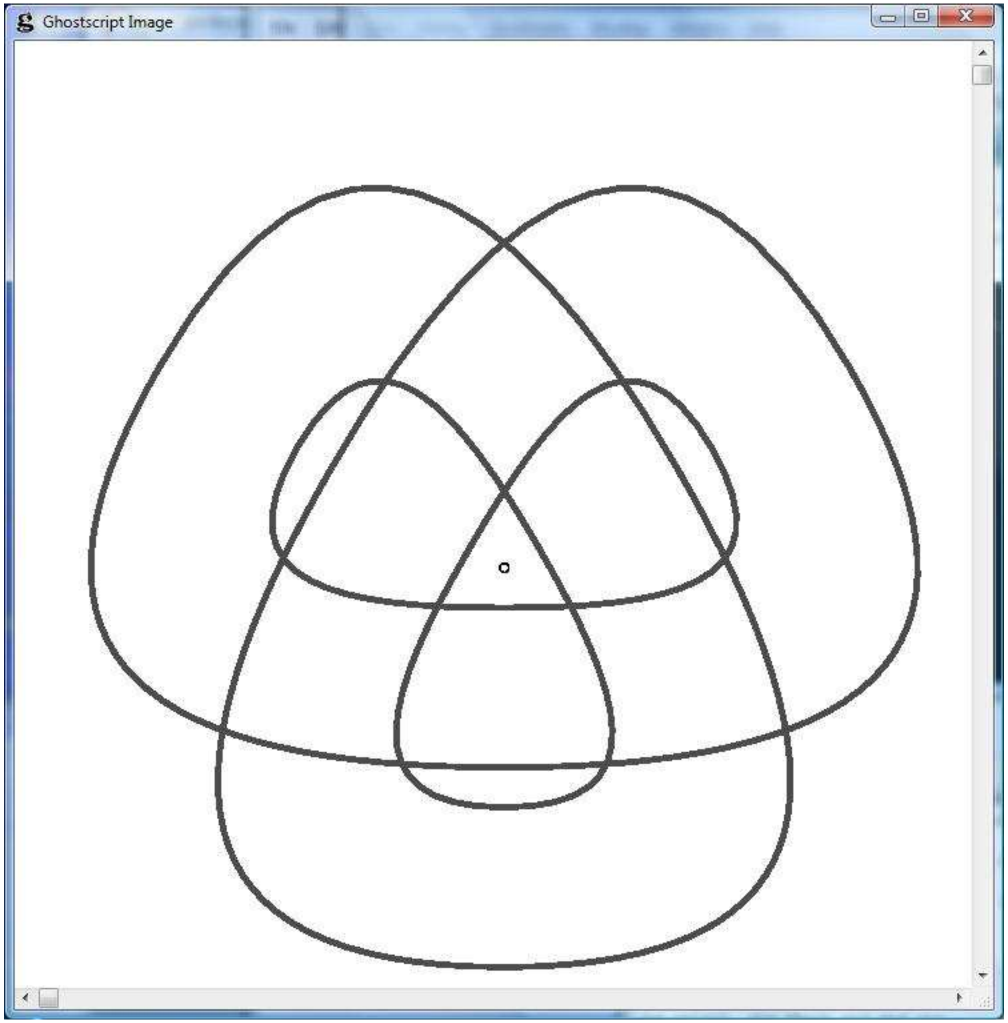
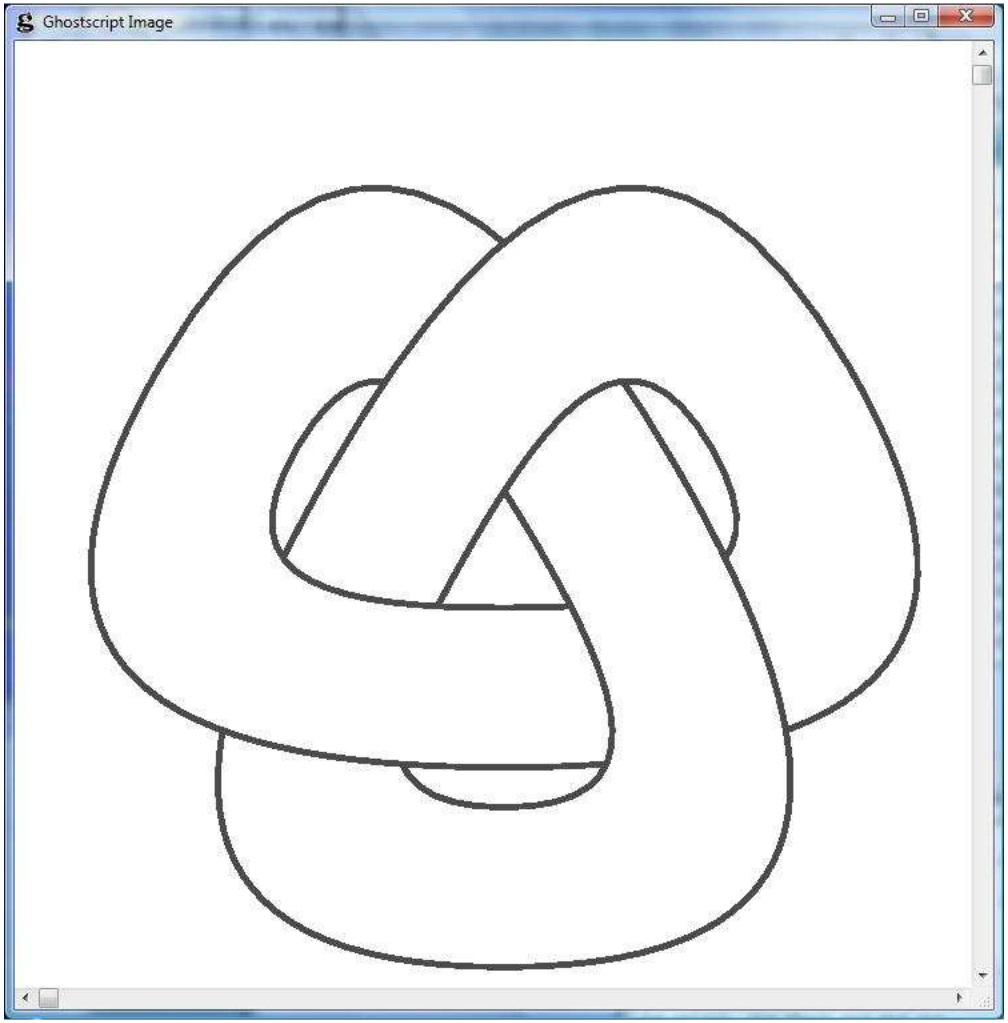
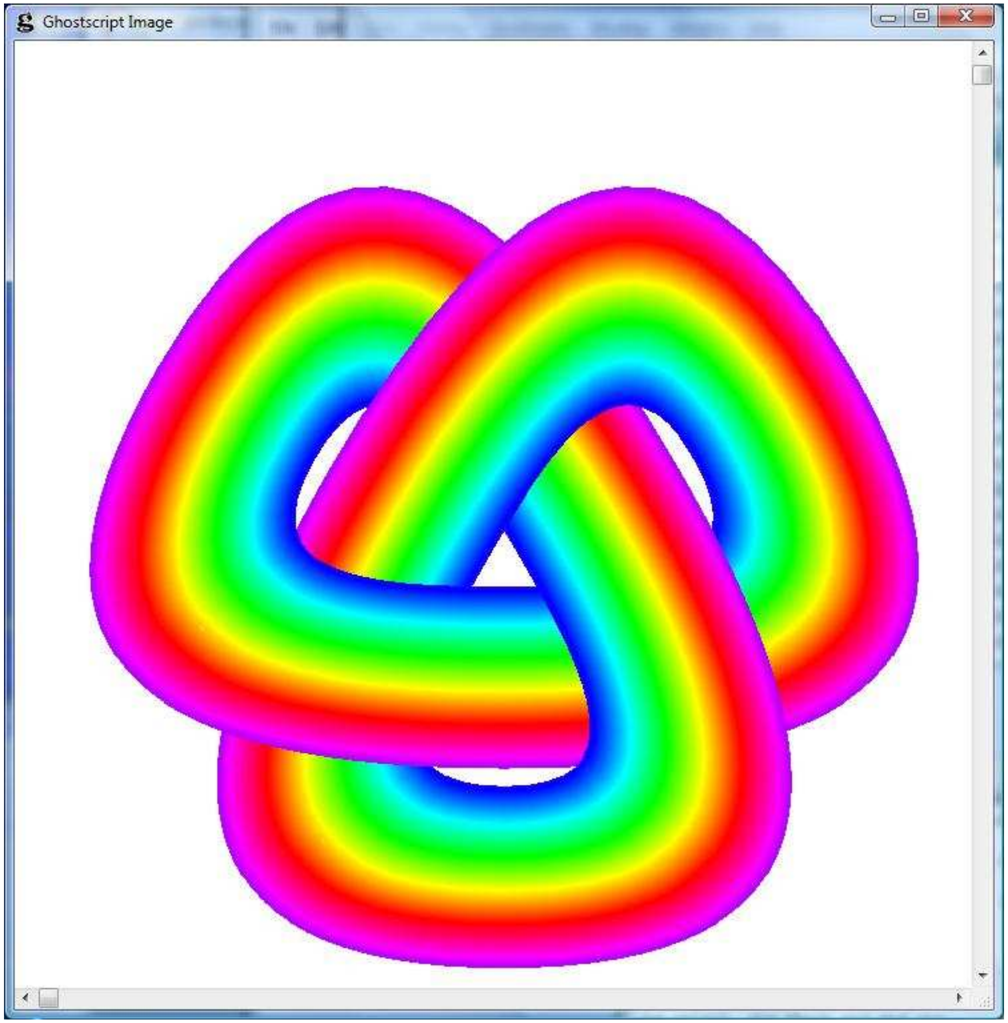


We do the same for the analogous points between $0.6\cdot(-3,\ \sqrt{3})$ and $0.1\cdot(1.5,\ \tfrac{1}{2}\sqrt{3})$. And then we'll rotate ... once ... and twice ... and then reflect across the y-axis. Now let's clean up a little ... There! Only ...



We had to “fill” the zones between outer and inner Bézier curves with (opaque) white “paint” in order that the unreflected arms of the trefoil not shine through the reflected ones. Now it’s more satisfying.



Or, the same thing in rainbow colors ... .

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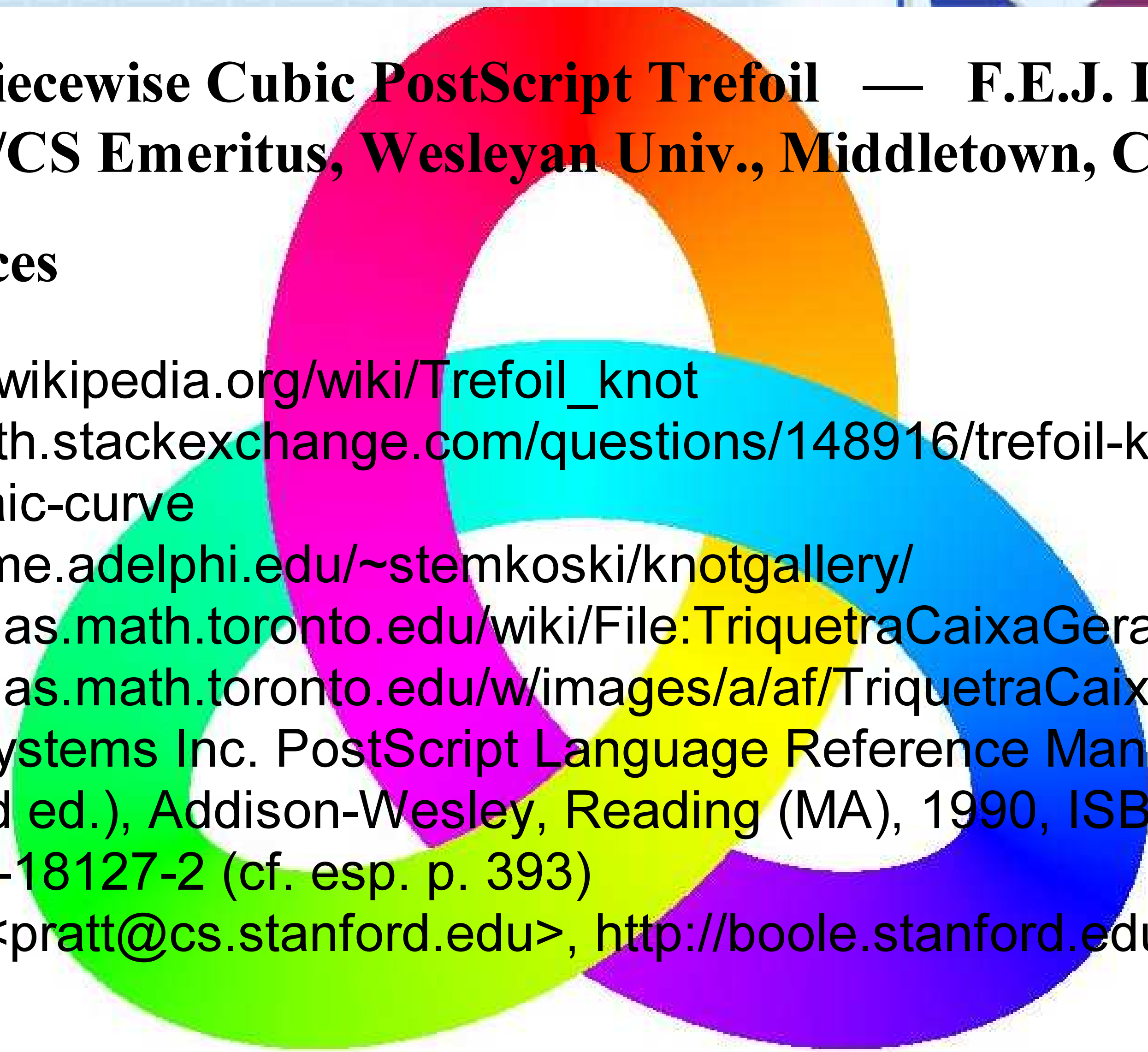
A Piecewise Cubic PostScript Trefoil — F.E.J. Linton Math/CS Emeritus, Wesleyan Univ., Middletown, CT, USA



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