CLASS NOTES OF PROF. LINTON'S LECTURES ON CATEGORY THEORY (DRAFT)

Fall 1996

Wesleyan University

Contents

LECTURE 1

A category "is" a matrix-monoid (9/9)

Some key concepts. Let us start by reviewing three important concepts that will apppear recurrently in the future.

PARTLY-ORDERED SETS (POSETS). Let X be a set with relation $\leq \subseteq X \times X$. A *poset* is a pair $(X, \leq)$ subject to the conditions

$$a \leq a$$
$$a \leq b \wedge b \leq c \Rightarrow a \leq c$$

for all elements $a, b, c \in X$. (Optional, *rarely* really wanted: $\forall a, b \in X : a \leq b \wedge b \leq a \Rightarrow a = b$.)

MONOIDS. Let X be a set with $e \in X$ and binary operation $* : X \times X \longrightarrow X$ that satisfies the equations:

$$e * x = x = x * e$$
$$x * (y * z) = (x * y) * z$$

I-INDEXED FAMILIES. Let $A : I \longrightarrow T$ be a function, and write $A(i) = A_i \in T$, hence $A : I \longrightarrow T$ is nothing else but $\{A_i\}_{i \in I}$. Consider a function $a : \mathcal{F} \longrightarrow I$. Notice the correspondence between both presentations:

$$\{A_i\}_{i \in I} \; \underset{\mathcal{F} = \dot{\bigcup} A_i}{\overset{T = \mathcal{P}(\mathcal{F}),\ A_i = a^{-1}(i)}{\Longleftarrow\!\!\Longrightarrow}} \; \mathcal{F} \xrightarrow{a} I$$

Square-matrix monoids. In the case when $I = R \times C$, an I-indexed family becomes a "matrix", and if $R = C$, a square matrix.

$$R \times C \xrightarrow[i,j \in C = R]{A} T \qquad\qquad \begin{array}{c} \mathcal{F} = \dot{\bigcup} A_{ij} \\ \downarrow \\ C \times C \\ \alpha \downarrow \;\; \omega \downarrow \\ C \end{array}$$

What is the usual "product" of matrices (square or not)?

$$(A \bullet B)_{ik} = \sum_{j \in J} A_{ij} \cdot B_{jk}$$

Here A is an $R \times J$ matrix and B is an $J \times C$ matrix ($i \in R, k \in C$). The usual interpretation is: The entries A_{ij}, B_{jk} are real (or complex) numbers, or elements of some (commutative) ring, $\sum$ is the summation, $\cdot$ is the multiplication. What if the entries are *other* things, with meaningful, relevant definitions for $\sum$ and $\cdot$? And for which such contexts can it make sense to ask that a square matrix A (with $R = C$) "act like" a monoid, with associative, unitary $A \bullet A \longrightarrow A$?

What arises as square-matrix monoids (multiplicative graphs, categories) when entries are required to be 0 or 1?, positive extended real numbers?, sets ? The following table shows the answer in each case:

matrix entries are	*what arises*
0 or 1	posets
pos. ext. real numbers	non-symm., non-sep. metric spaces
sets	categories

The details are as follows.

POSETS. Define the multiplication (recall we have only 0's and 1's) as the usual multiplication on the reals, and the summation as the maximum: $\sum_k s_k =_{def} \max_k\{s_k\}$. Also, if $A_{ij} = 1$ write $i \leq j$, and one thinks "map from ϵ to δ" (for $\epsilon = 0, 1,\ \delta = 0, 1$) if and only if $\epsilon \leq \delta$.

$$(A \bullet A)_{ij} = \sum_k A_{ik} A_{kj}$$
$$(A \bullet A)_{ij} \longrightarrow A_{ij}$$

CATEGORIES. Remember that now the entries of the matrix are sets. The multiplication here is the cartesian product, and the summation is the disjoint union. So, a square matrix A ($C \times C$) is a bunch of sets and

A_{ij} "=" a collection of maps getting to i from j.
"=" a bunch of proofs of i from hypothesis j.
"=" ...

Here

$$(A \bullet A)_{ij} = \sum_k (A_{ik} \cdot A_{kj}) = \cup_k (A_{ik} \times A_{kj})$$

$$\begin{tikzcd}
 & (A \bullet A)_{ij} \arrow[d, "\text{comp.}"] \\
\begin{pmatrix} 1 & 0 & 0 \\ & \ddots & \\ 0 & 0 & 1 \end{pmatrix} \arrow[ur, "\cancel{e_i \in A_{ii}}"] \arrow[r, "e"'] & (A)_{ij}
\end{tikzcd}$$

[not necessarily SYMMETRIC] METRIC SPACES. Here the entries are non-negative real numbers extended with ∞. The multiplication is ordinary arithmetic addition. Finally the summation is the infimum: $\sum_k = \inf\{s_k\}$.

$$(A \bullet A)_{ij} = \sum_k (d_{ik} \cdot d_{kj}) = \inf_k \{d(i,k) + d(k,j)\}$$

and one thinks "map from r to s" if and only if $r \geq s$.

$$\begin{tikzcd}
 & (A \bullet A)_{ij} \arrow[d, "\Delta - \text{ineq.}"] \\
\begin{pmatrix} 0 & & \infty \\ & \ddots & \\ \infty & 0 & 0 \end{pmatrix} \arrow[ur, "\cancel{}"] \arrow[r, "d(i,i) \leq 0"'] & (A)_{ij}
\end{tikzcd}$$

REMARK 1. There are generalizations of these three constructions, namely U-promonoidal categories, where the "domain" U of matrix entries is (the class of objects of) a "closed", or a "monoidal", or a "promonoidal" category. But we are getting way ahead of ourselves; see Lecture 26.

LECTURE 2

Categories (9/11)

DEFINITION 1. A *category* $\mathcal{A}$ (with object-class $|\mathcal{A}|$, hom-sets $\mathcal{A}(A, B)$, composition rule $\circ$, and identity maps e_A) is given by specifying:

1. A class $|\mathcal{A}|$ of *objects*.
2. For each $A, B \in |\mathcal{A}|$, a set $\mathcal{A}(A, B)$ (also written $\hom_{\mathcal{A}}(A, B)$ or $\mathrm{mor}_{\mathcal{A}}(A, B)$) of *morphisms* (maps, arrows) from A to B.
3. For each $A, B, C \in \mathcal{A}$, a function

$$\mathcal{A}(B, C) \times \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, C) \quad :: (f, g) \mapsto f \circ_{\mathcal{A}} g$$

4. For each $A \in \mathcal{A}$, a distinguished map $e_A \in \mathcal{A}(A, A)$ (also usually denoted by id_A).

all subject to the following conditions:

(5) For all $A, B, C, D \in \mathcal{A}$,
for all $f \in \mathcal{A}(C, D)$, $g \in \mathcal{A}(B, C)$, $h \in \mathcal{A}(A, B)$:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

(6) For all $A, B, C \in \mathcal{A}$,
for all $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$:

$$g \circ e_B = g \qquad e_B \circ f = f$$

REMARK 2. Some authors require that the hom-sets be disjoint. I have not found any good reason to put in that restriction. I think that this presentation allows more flexibility.

Usually categorical concepts are expressed by means of diagrams:

$$\text{write} \quad A \xrightarrow{f} B \quad \text{for} \quad f \in \mathcal{A}(A, B)$$

So, axiom (5) could be stated as:

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[dr, "g \circ h"'] \arrow[drr, bend left, "(f\circ g)\circ h"] \arrow[drr, bend right, "f\circ(g\circ h)"'] & B \arrow[d, "g"] \arrow[dr, "f \circ g"] & \\
& C \arrow[r, "f"'] & D
\end{tikzcd}$$

and axiom (6) states that in the following diagram each path gives the same result:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "f"'] & B \arrow[d, "e_B"] \arrow[dr, "g"] & \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

$$\begin{tikzcd}
& B \arrow[d, "e_B"'] \arrow[dr, "g \circ e_B = g"] & \\
A \arrow[r, "f"] \arrow[dr, "e_B \circ f = f"'] & B \arrow[r, "g"] \arrow[d, "e_B"'] & C \\
& B &
\end{tikzcd}$$

Examples.

Sets. The category $\mathcal{S}ets$. Here $|\mathcal{S}ets|$ is *some* universe of sets (your favorite model), and $\mathcal{S}ets(A, B) = B^A$. The composition rule is the usual composition of functions.

Topological Spaces. The category $\mathcal{T}op$. Here $|\mathcal{T}op|$ denotes all topological spaces and $\mathcal{T}op(A, B)$ all continuous functions from A to B.

Groups. The category here is $\mathcal{G}p$. $|\mathcal{G}p|$ is all the groups. On the other side $\mathcal{G}p(A, B)$ denotes all group homomorphisms from A to B.

Abelian Groups. $\mathcal{A}bGp$ is similar as $\mathcal{G}p$, but about commutative groups.

Modules. The category Λ-$\mathcal{M}od$ is defined for Λ a (commutative) ring. Λ-$\mathcal{M}op(A, B)$ is the collection of Λ-homomorphisms.
d

MONOIDS. If $(M, \cdot, e_M)$ is a monoid (with multiplication $x \cdot y$ written just xy, and unity e_M), let $\mathcal{M}_M$ be the category defined by:

$$\begin{aligned} |\mathcal{M}_M| &= 1, \quad \text{where } 1 = \{*\} \\ \mathcal{M}_M(*, *) &= M \\ x \circ_M y &= xy \\ e_* &= e_M \end{aligned}$$

DISCRETE CATEGORY. Fix a set X. Define the category $\mathcal{D}_X$ by

$$|\mathcal{D}_X| = X$$

$$\mathcal{D}_X(x, y) = \begin{cases} \emptyset & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}$$

$\mathcal{D}_X$ is called the *discrete* category on the class of objects X.

POSETS. If $(P, \leq)$ is a partial ordered set (poset), define $\mathcal{C}_P$ by

$$|\mathcal{C}_P| = P$$

$$\mathcal{C}_P(x, y) = \{(x, y)\} \cap \leq \; = \begin{cases} \emptyset & \text{if } x \not\leq y \\ \{(x, y)\} & \text{if } x \leq y \end{cases}$$

The hom-set can be alternatively defined as

$$\mathcal{C}_P(x, y) = \begin{cases} \emptyset & \text{if } x \not\leq y \\ 1 & \text{if } x \leq y \end{cases}$$

From this point of view, $\mathcal{D}_X$ is nothing else than the category $\mathcal{D}_{(X, \leq)}$, where $\leq$ is the discrete order in X.

NATURAL NUMBERS. Using the examples of Monoids, Discrete and Poset Categories above, we can define:

1. $\mathbf{0}$: the empty category.
2. $\mathbf{1}$: the 1-object, 1-morphisms category.
3. In general, for each ordinal number n, $\mathbf{n} = \mathcal{C}_n$.
4. Also we can define alternatively $\mathbf{2}$ as the category

$$\bullet_0 \xrightarrow{\leq} \bullet_1$$

5. $\mathbf{3}$ as the category

$$\begin{tikzcd} & \mathbf{1} \arrow[dr] & \\ \mathbf{0} \arrow[ur] \arrow[rr] & & \mathbf{2} \end{tikzcd}$$

6. **4** as the category

$$\begin{tikzcd}
 & \mathbf{1} \arrow[dd] \arrow[dr] & \\
\mathbf{0} \arrow[ur] \arrow[rr] \arrow[dr] & & \mathbf{3} \\
 & \mathbf{2} \arrow[ur] &
\end{tikzcd}$$

7. An so on: **n** would be the oriented graph with n vertices, and edges from i to j for each $i \leq j$, $i, j \in n$.
8. $\mathcal{M}_{(\mathbb{N},+)}$: It is difficult to draw it here: is a $\bullet$ with one circular arrow around it for each natural number.

Note that we have at least 3 different categories that represent the natural numbers: $\mathcal{D}_{\mathbb{N}}$ (the discrete category), $\mathcal{C}_{\mathbb{N}}$ (with the order relation of $\mathbb{N}$), and $\mathcal{M}_{(\mathbb{N},+)}$ (with the monoidal structure of $\mathbb{N}$).

Δ-ALGEBRAS. One last, and important, (family of) example(s) of (a type of) category: Δ-$\mathcal{A}lg$. Fix a set Δ and a function $n : \Delta \longrightarrow |\mathcal{S}ets|$. By a (lawless[1]) Δ-algebra (better would be: "(Δ, n)-algebra") is meant a set X together with one honest-to-god function

$$X_\omega : X^{n(\omega)} \longrightarrow X$$

for each $\omega \in \Delta$. We define $|\Delta\text{-}\mathcal{A}lg|$ as all Δ-algebras, and the arrows of this category, $\Delta\text{-}\mathcal{A}lg((X, (X_\omega)_{\omega\in\Delta}), (Y, (Y_\omega)_{\omega\in\Delta}))$, by

$$\{f \in Y^X \mid \forall \omega \in \Delta, \text{ diagram (1) commutes}\}$$

(1)
$$\begin{tikzcd}
X^{n(\omega)} \arrow[r, "(f)"] \arrow[d, "X_\omega"'] & Y^{n(\omega)} \arrow[d, "Y_\omega"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

It is easy to see that the composition works, checking the commutativity of the outside box in the following diagram (this means that you can go 'by different paths' in the diagram):

$$\begin{tikzcd}
X^{n(\omega)} \arrow[r, "(f)"] \arrow[d, "X_\omega"'] & Y^{n(\omega)} \arrow[r, "(g)"] \arrow[d, "Y_\omega"'] & Z^{n(\omega)} \arrow[d, "Z_\omega"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Some Constructions.

[1]We have ~~still~~ no way yet of ~~expressing~~ imposing equations (or laws)!

THE POSET OF A CATEGORY. If $\mathcal{X}$ is a category, write $PO_{\mathcal{X}} = |\mathcal{X}|$, and define the relation $\leq$ in $PO_{\mathcal{X}}$ as:

$$A \leq_{PO_{\mathcal{X}}} B \text{ iff there is } f \in \mathcal{X}(A, B)$$

So, we have that $A \leq_{PO_{\mathcal{X}}} A$ because $e_{\mathcal{X}} \in \mathcal{X}(A, A)$, and if $A \leq_{PO_{\mathcal{X}}} B$ and $B \leq_{PO_{\mathcal{X}}} C$, then $A \leq_{PO_{\mathcal{X}}} C$ by property (4) of the definition of category. So, to every category $\mathcal{X}$ we can associate a poset $PO_{\mathcal{X}}$. It can be easily checked that $PO_{\mathcal{D}_{(X,\leq)}} = (X, \leq)$.

THE OPPOSITE CATEGORY. If $\mathcal{A}$ is a given category, the *opposite* category of $\mathcal{A}$, denoted by $\mathcal{A}^{op}$ is given as follows:

$$|\mathcal{A}^{op}| = |\mathcal{A}|$$
$$\mathcal{A}^{op}(A, B) = \mathcal{A}(B, A) \qquad f \circ_{\mathcal{A}^{op}} g = g \circ_{\mathcal{A}} f$$

It is important to notice that $(\mathcal{A}^{op})^{op} = \mathcal{A}$. The classical model for this construction is the reversing of the ordering of posets, and in non-commutative rings and groups by reversing the operation $x \cdot_{op} y = y \cdot x$).

Generally, $\mathcal{A}$ and $\mathcal{A}^{op}$ will have very different properties, and will not resemble each other at all, e.g.

$\mathcal{S}ets$	$\mathcal{S}ets^{op}$
$\emptyset$ is a universal source	$\emptyset$ is a universal target
$\mathbf{1}$ is a universal target	$\mathbf{1}$ is a universal source
$\forall x (x \neq \emptyset \vee \exists\, f : \mathbf{1} \longrightarrow x)$	false here!

Yet *sometimes* $\mathcal{A}$ and $\mathcal{A}^{op}$ may be virtually identical, as with locally compact abelian topological groups, according to Pontriagin duality.

THE PRODUCT CATEGORY. Given categories $\mathcal{A}$ and $\mathcal{B}$, define their *product* by

$$|\mathcal{A} \times \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$$
$$\{\mathcal{A} \times \mathcal{B}\}((A_1, B_1), (A_2, B_2)) = \mathcal{A}(A_1, A_2) \times \mathcal{B}(B_1, B_2)$$

The composition of arrows works in the obvious way:

$$\begin{tikzcd}
(A_1 \arrow[d, "f_1"'] \arrow[dd, dashed, bend right=60, "g_1 \circ f_1"'] & , & B_1) \arrow[d, "f_2"] \arrow[dd, dashed, bend left=60, "g_2 \circ f_2"] \\
(A_2 \arrow[d, "g_1"'] & , & B_2) \arrow[d, "g_2"] \\
(A_3 & , & B_3)
\end{tikzcd}$$

Notice how *one* large family of categories $\mathcal{X}$ looking just like $\mathcal{X}^{op}$ is given by $\mathcal{X} = \mathcal{A} \times \mathcal{A}^{op}$. For, generally,

$$(\mathcal{A} \times \mathcal{B})^{op} = \mathcal{A}^{op} \times \mathcal{B}^{op}$$

So, in particular

$$(\mathcal{A} \times \mathcal{A}^{op})^{op} = \mathcal{A}^{op} \times (\mathcal{A}^{op})^{op} = \mathcal{A}^{op} \times \mathcal{A}$$

and it is not hard (using $(x, y) \mapsto (y, x)$) to make $\mathcal{X}^{op} = \mathcal{A}^{op} \times \mathcal{A}$ "look just like" $\mathcal{A} \times \mathcal{A}^{op} = \mathcal{X}$.

THE "SLICE" CATEGORY. Given a category $\mathcal{A}$, for $I \in \mathcal{A}$ define the category $\mathcal{A}|_I$ ("$\mathcal{A}$ slice(d over) I") by:

$$|\mathcal{A}|_I| = \dot{\bigcup}_{A \in \mathcal{A}} \mathcal{A}(A, I)$$

$$\mathcal{A}|_I \left(\begin{array}{c} A \\ {\scriptstyle F}\big\downarrow \\ I \end{array} , \begin{array}{c} B \\ \big\downarrow{\scriptstyle G} \\ I \end{array} \right) = \{ \text{ all } A \xrightarrow{\varphi} B \mid G \circ \varphi = F \}$$

that is, the objects are arrows with target I, and the arrows are maps $A \xrightarrow{\varphi} B$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "F"'] & & B \\ & I \arrow[ur, "G"'] & \end{tikzcd}$$

LECTURE 3

Functors (9/16)

We will start with an example. Consider the "transformation" U from the category Δ-$\mathcal{A}lg$ to $\mathcal{S}ets$, sending each Δ-algebra to its underlying set, and a Δ-morphism to the set-function underlying it.

$$\begin{tikzcd} \Delta\text{-}\mathcal{A}lg \arrow[d, "U"'] & (X, \{X_\omega\}_{\omega\in\Delta}) \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, ""] & (Y, \{Y_\omega\}_{\omega\in\Delta}) \arrow[d] \\ \mathcal{S}ets & X \arrow[r, "f"] & Y \end{tikzcd}$$

The composite can be done in the natural way. As you see, things are the same. We are just looking them in a different framework. This is just a particular example of a more general concept:

DEFINITION 2. Given categories $\mathcal{X}, \mathcal{A}$, by a *functor* from $\mathcal{X}$ to $\mathcal{A}$ is meant any couple of rules

$$F : |\mathcal{X}| \longrightarrow |\mathcal{A}|$$
$$\forall X, Y \in \mathcal{X} \quad F : \mathcal{X}(X,Y) \longrightarrow \mathcal{A}(FX, FY)$$

meeting the following requirements:

$$F(g \circ f) = F(g) \circ F(f)$$
$$F(e_X) = e_{F(X)}$$

We can see that our example above is a functor, called U_Δ. In the same way that small categories helped us understand some basic ideas, functors between small categories help clarify concepts too.

EXAMPLE 1. Functors $F : \mathbf{1} \longrightarrow \mathcal{X}$. Here we have $F(*) \in \mathcal{X}$ and $F(e_*) = e_F(*)$. Notice the correspondence:

$$\begin{array}{ccc} \mathbf{1} \xrightarrow{F} \mathcal{X} & \Longrightarrow & F(*) \in |\mathcal{X}| \\ * \longrightarrow A & \Longleftarrow & A \in \mathcal{X} \end{array}$$

i.e. a functor $F : \mathbf{1} \longrightarrow \mathcal{X}$ "is an object" of $\mathcal{X}$.

Example 2. Functors $F : \mathbf{2} \longrightarrow \mathcal{X}$. Such a functor fits in the following diagram:

$$\begin{tikzcd}
\mathbf{1} \arrow[dr, "\ulcorner 0 \urcorner"'] \arrow[drr, "F\circ\ulcorner 0\urcorner = \ulcorner A \urcorner"] & & \\
& \mathbf{2} \arrow[r, "\ulcorner \alpha : A \to B \urcorner"] & \mathcal{X} \\
\mathbf{1} \arrow[ur, "\ulcorner 1 \urcorner"] \arrow[urr, "F\circ\ulcorner 1\urcorner = \ulcorner B \urcorner"'] & &
\end{tikzcd}$$

and therefore "determines" two objects, $A = F(0)$ and $B = F(1)$ of $\mathcal{X}$. But wait: there is more. What is this functor? It is just a map $A \xrightarrow{\alpha} B$ in $\mathcal{X}$ arising as $\alpha = F(0 \leq 1)$ (with label, source and target). In some sense, "$\mathbf{2}$" is the platonic version of a map, and $A \xrightarrow{\alpha} B$ its concrete realization.

Now consider the following diagram:

$$\begin{tikzcd}
\mathcal{X} \arrow[dr, "(\mathrm{id},0_{\mathrm{id}_0})"'] \arrow[drr, "F"] & & \\
& \mathcal{X} \times \mathbf{2} \arrow[r, "\theta"] & \mathcal{A} \\
\mathcal{X} \arrow[ur, "(\mathrm{id},1_{\mathrm{id}_1})"] \arrow[urr, "G"'] & &
\end{tikzcd}$$

Take $X \xrightarrow{\alpha} Y$ in $\mathcal{X}$. We have the following diagram:

$$\begin{tikzcd}
F(X) = \theta(X,0) \arrow[r] \arrow[d, "F(\alpha) = \theta(\alpha, id_0)"'] & G(X) = \theta(X,1) \arrow[d, "G(\alpha) = \theta(\alpha, id_1)"] \\
F(Y) = \theta(Y,0) \arrow[r] & G(Y) = \theta(Y,1)
\end{tikzcd}$$

But even more. For each $X \in \mathcal{X}$:

$$\theta(e_X, {}_0\!\leq_1) = \theta_X : F(X) \longrightarrow G(X)$$

N.a.s.c. that the data $F(X), G(X), F(\alpha), G(\alpha), \theta_X$ as above that arise from the functor $\theta : \mathcal{X} \times \mathbf{2} \longrightarrow \mathcal{A}$ are:

1. F define a functor $\mathcal{X} \longrightarrow \mathcal{A}$.
2. G define also a functor $\mathcal{X} \longrightarrow \mathcal{A}$.
3. The various maps $\theta_X : F(X) \longrightarrow G(X)$ fulfill:

$$\text{For all } X, Y \in \mathcal{X}, \text{ for all } \alpha \in \mathcal{X}(X,Y):$$

(2)
$$\begin{tikzcd}
F(X) \arrow[r, "\theta_X"] \arrow[d, "F(\alpha)"'] \arrow[dr, dotted, "\theta(\alpha,\leq)"] & G(X) \arrow[d, "G(\alpha)"] \\
F(Y) \arrow[r, "\theta_X"] & G(Y)
\end{tikzcd}$$

diagram (2) commute.

It is clear that F and G are functors. The commutativity of diagram (2) follows from the three ways we can see the functor $(\alpha, \leq)(X_0, \mathbf{0} \longrightarrow (Y_1, \mathbf{1})$, which is the diagonal in diagram (2)

$$\begin{tikzcd}
 & (X_0, \mathbf{1}) \arrow[dr, "(\alpha,\mathrm{id}_1)"] & \\
(X_0, \mathbf{0}) \arrow[ur, "(\mathrm{id}_X,\leq)"] \arrow[rr, "(\alpha,\leq)"] \arrow[dr, "(\alpha,\mathrm{id}_0)"'] & & (Y_1, \mathbf{1}) \\
 & (Y_1, \mathbf{0}) \arrow[ur, "(\mathrm{id}_1,\leq)"'] &
\end{tikzcd}$$

It is easy to see that the converse holds too, that is if we have conditions (1),(2) and (3) above, we have a functor from $\mathbf{2}$ to $\mathcal{X}$. This is what does it mean to have a natural transformation from F to G. So we have now two viewpoints to see natural transformations.

Definition 3. Given functors $F, G : \mathcal{X} \longrightarrow \mathcal{A}$, a *natural transformation* θ from F to G "is" a family $\{\theta_X\}_{X \in \mathcal{X}}$ of $\mathcal{A}$-morphisms $\theta_X : F(X) \longrightarrow G(X)$ such that diagram (2) commutes.

Or, in other presentation, a natural transformation between functors $F, G : \mathcal{X} \longrightarrow \mathcal{A}$ is a functor

$$\theta : \mathcal{X} \times \mathbf{2} \longrightarrow \mathcal{A}$$

such that composing with

$$\begin{tikzcd}
\mathcal{X} \arrow[dr, "(\mathrm{id}_*,0)"] & \\
 & \mathcal{X} \times \mathbf{2} \\
\mathcal{X} \arrow[ur, "(\mathrm{id}_*,1)"'] &
\end{tikzcd}$$

to give exactly F and G.

Category of Categories. Given categories $\mathcal{A}$ and $\mathcal{B}$, appropriate "maps" from $\mathcal{A}$ to $\mathcal{B}$ are the functors $F : \mathcal{A} \longrightarrow \mathcal{B}$. We can build the category $\mathcal{C}at$, defined so:

$$|\mathcal{C}at| = \{\text{ "all" categories}\}$$
$$\mathcal{C}at(\mathcal{A}, \mathcal{B}) = \{F : \mathcal{A} \longrightarrow \mathcal{B} \mid F \text{ functor }\}$$

Composition is defined in the natural way. This is a category except ... it makes no sense for set theory.

Category of Functors. Now, considering $\mathcal{C}at(\mathcal{A}, \mathcal{B})$ as objects, we can form a new category, $\mathrm{Func}(\mathcal{A}, \mathcal{B})$, whose arrows are natural transformations:

$$|\mathrm{Func}(\mathcal{A}, \mathcal{B})| = \mathcal{C}at(\mathcal{A}, \mathcal{B})$$
$$\{\mathrm{Func}(\mathcal{A}, \mathcal{B})\}(F, G) = \text{n.t.}(F, G)$$

Given three functors $F, G, H : \mathcal{X} \longrightarrow \mathcal{A}$, and natural transformations $F \overset{\theta}{\Rightarrow} G$ and $G \overset{\lambda}{\Rightarrow} H$ as in the diagram below

$$\begin{tikzcd}[column sep=huge] \mathcal{X} \arrow[r, bend left=40, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=40, "H"'{name=H}] & \mathcal{A} \arrow[from=F, to=G, phantom, "\Downarrow\theta"] \arrow[from=G, to=H, phantom, "\Downarrow\lambda"] \end{tikzcd}$$

we have for each $X \in \mathcal{X}$:

$$F(X) \xrightarrow{\theta_X} G(X) \xrightarrow{\lambda_X} H(X)$$

and it is not hard to verify that the family $\{(\lambda \circ \theta)_X\}_{X \in \mathcal{X}}$ defined as

$$(\lambda \circ \theta)_X = \lambda_X \circ \theta_X$$

is a natural transformation.

Can we continue?. Once again, consider $\{\mathrm{Func}(\mathcal{A}, \mathcal{B})\}(F, G)$ and take two natural transformations (as objects):

$$\begin{tikzcd}[column sep=huge] \mathcal{X} \arrow[r, bend left=30, "F"{name=F}] \arrow[r, bend right=30, "G"'{name=G}] & \mathcal{A} \arrow[from=F, to=G, phantom, "\theta_1\Downarrow \overset{?}{\Rightarrow} \Downarrow\theta_2"] \end{tikzcd}$$

Question to sharpen mind's teeth: What would be $\overset{?}{\Rightarrow}$? (i.e. an arrow between natural transformations?)

Functor in two variables. Let us generalize the construction in 2. Take a functor

$$\mathcal{X} \times \mathcal{Y} \xrightarrow{T} \mathcal{A}$$

and write

$$T_{X_1} = T(X_1, -) : \mathcal{Y} \longrightarrow \mathcal{A}$$
$$T^{X_2} = T(-, X_2) : \mathcal{X} \longrightarrow \mathcal{A}$$

Recall T^0 and T^1 were F and G respectively in example 2. Is there some side-condition we can impose so that data of this sort to be arising from T_{X_0} and T^{X_1} ? The answer is yes (and exactly the same condition): For all $(X_0 \overset{\alpha}{\to} X_1) \in \mathcal{X}$, for all $(Y_0 \overset{\beta}{\to} Y_1) \in \mathcal{Y}$:

(3)
$$\begin{tikzcd}
T(X_0, Y_0) \arrow[r, "T_{X_0}(\beta)"] \arrow[d, "T^{Y_0}(\alpha)"'] & T(X_0, Y_1) \arrow[d, "T^{Y_1}(\alpha)"] \\
T(X_1, Y_0) \arrow[r, "T_{X_1}(\beta)"] & T(X_1, Y_1)
\end{tikzcd}$$

should commute. The reason is exactly the same as in the special case of 2 (there $\mathcal{Y}$ was $\mathbf{2}$ and β was id_0 or id_1 or $\leq$).

$$\begin{tikzcd}
& (X_0, Y_1) \arrow[dr, "(\alpha,\mathrm{id})"] & \\
(X_0, Y_0) \arrow[ur, "(\mathrm{id},\beta)"] \arrow[rr, "(\alpha,\beta)"] \arrow[dr, "(\alpha,\mathrm{id})"'] & & (X_1, Y_1) \\
& (X_1, Y_0) \arrow[ur, "(\mathrm{id},\beta)"'] &
\end{tikzcd}$$

You could hope that maybe a function $f : X \times Y \longrightarrow Z$ is continuos iff each f_X and f^Y are continuous, plus some compatibility condition replacing the commutativity of a diagram like (3). Unfortunately no such translation of condition (3) seems possible or meaningful in this context.

LECTURE 4

Δ-Algebras (9/18)

Consider the following diagram defining $U_\Delta^n(X) = (U_\Delta(X))^n$:

$$\begin{tikzcd}
\Delta\text{-}\mathcal{A}lg \arrow[d, "U_\Delta"'] \arrow[dr, "(U_\Delta)^n"] & \\
\mathcal{S}ets \arrow[r, "(\;)^n", "\mathcal{S}ets(n,-)"'] & \mathcal{S}ets
\end{tikzcd}$$

What should be understood by $\mathrm{nat}(U_\Delta^n, U_\Delta)$? Does $\mathrm{nat}(U_\Delta^n, U_\Delta)$ form a Δ-algebra? When a ring-theorist, a group-theorist think what are the "natural" operations for a ring, for a group, they think in operations of certain arity, such $+$, $*$, an so on. The "natural" operations in a lattice include both joins, meets, etc.

NATURAL OPERATIONS. By a *natural* operation λ on Δ-algebras (of *arity* n (n a set)) is meant a scheme of some sort by which, for each Δ-algebra $(X, (X_\omega)_{\omega\in\Delta})$ and each n-tuple $x = (\cdots x_i \dots)_{i\in n}$ from X, that is $x \in X^n$, there is associated in a "natural way" a value $\lambda(x) \in X$. A natural way such that whenever $X \xrightarrow{f} Y$ is already a Δ-algebra homomorphism, then

$$f(\lambda(\cdots x_i \cdots)_{i\in n}) = \lambda(\cdots f(x_i) \cdots)_{i\in n}$$

(every Δ-algebra must "preserve" this operation), i.e.

$$\boxed{\lambda \textit{ is just a natural transformation } U_\Delta^n \longrightarrow U_\Delta}$$

Let us present a very 'natural' collection of natural transformations:

$$\begin{aligned} g\ : n &\longrightarrow \mathrm{nat}(U_\Delta^n, U_\Delta) \\ i &\mapsto g_i \end{aligned}$$

where the family $(g_i : U_\Delta^n \longrightarrow U_\Delta)_X$ is just the family of projections[1]:

$$\begin{aligned}(g_i)_X : (U_\Delta(X))^n &\longrightarrow U_\Delta(X)\\ (x_1, \ldots, x_n) &\mapsto x_i\end{aligned}$$

(Note that for algebras with more than two elements, $g_i \neq g_j$ for $i \neq j$.)

Another collection of natural transformations is the following one, given by Δ: Given $\omega \in \Delta$, we can find

$$\lambda_\omega : (U_\Delta)^{n(\omega)} \longrightarrow U_\Delta$$

defined (let us write ω instead of λ_ω to simplify notations) by

$$\begin{aligned}&(\lambda_\omega)_X : |X|^{n(\omega)} \longrightarrow |X| = U_\Delta^{n(\omega)}(X) \longrightarrow U_\Delta(X)\\ &= X_\omega : |X|^{n(\omega)} \longrightarrow |X|\end{aligned}$$

To check that this is a natural transformation we have to check the commutativity of the following diagram:

$$\begin{tikzcd} X^{n(\omega)} \arrow[r] \arrow[d, "X_\omega"'] & Y^{n(\omega)} \arrow[d, "Y_\omega"] \\ X \arrow[r, "f\ (\text{homo})"'] & Y \end{tikzcd}$$

Derived Operations. (Nonetheless) given $\omega \in \Delta$ and given $n(\omega)$ n-ary operations $\lambda_j \in \mathrm{nat}((U_\Delta)^n, U_\Delta)$, we wish to define $\omega((\cdots \lambda_j \cdots)_{j \in n(\omega)}) \in \mathrm{nat}((U_\Delta)^n, U_\Delta)$. So, for each Δ-algebra X, let us give the X^{th} component of $\omega(\lambda)$ as the composite:

$$\{\omega((\cdots \lambda_j \cdots)_{j \in n(\omega)})\}_X : |X|^n \xrightarrow{(\cdots(\lambda_j)_X \cdots)_{j\in n(\omega)}} |X|^{n(\omega)} \xrightarrow{X_\omega} X$$

Exercise. Why is this a natural transformation as indicated? (hint: has to do with the fact that the composition of natural transformations is a natural transformation).

Universal algebraists also speak of *derived* operations, or *polynomial* operations on Δ-algebras ... Those, of arity n are just the *natural* operations in the *subalgebra* of $\mathrm{nat}(U_\Delta^n, U_\Delta)$ generated by the previously described g_i $(i \in n)$. Certainly each original n-ary ω appears there (more accurately, what we called λ_ω does) as

$$\lambda_\omega = \omega((\cdots g_i \cdots)_{i \in n}$$

[1]The notation g_i to suggest the i^{th}-generator of a free group in group theory.

It is easy to check. Indeed, when $n = n(\omega)$,

$$|X|^n \overset{(\cdots(g_i)_X\cdots)_{i\in n}}{\longrightarrow} |X|^n = \mathrm{id}_{|X|^n}$$

so

$$(\lambda_\omega)_X = X_\omega \circ \mathrm{id} = X_\omega$$

Generated Subalgebra. Let Δ, $n : \Delta \longrightarrow |Sets|$ be given, let A be a (not necessarily small) Δ-algebra. Let $G \subseteq A$ be a subset of A, and let $r =$ the least regular cardinal wiht no $n(\omega)$ cofinal in it[2]. (This r exists if Δ is a set). Let us call r the *rank* of $(\Delta, (n(\omega))_{\omega\in\Delta})$. By a Δ-*subalgebra of A generated by* $G \subseteq A$, we mean the smallest subclass $S \subseteq A$ "closed under the operations $\omega \in \Delta$", that formally means:

$$\forall \bar{(a)} \in A^{n(\omega)} \forall \omega \in \Delta \forall i \in n(\omega)(a_i \in S \Rightarrow w(\bar{a}) \in S)$$

Crucial Claim: the upper bound for *size* of S is a simple set function of r and $\overline{\overline{G}}$.

Proof. (Due essentially to Baire: is just the Baire construction of Borel sets).

$$X_0 = G$$

$$X_1 = X_0 \cup \bigcup_{\omega\in\Delta} \omega(X_0^{n(\omega)}) \subseteq A$$

$$X_2 = X_1 \cup \bigcup_{\omega\in\Delta} \omega(X_1^{n(\omega)})$$

$$\vdots$$

Now, having X_λ for $\lambda < \beta$ define

$$x_\beta = \bigcup_{\lambda<\beta} X_\lambda \cup \bigcup_{\omega\in\Delta} \omega(\bigcup_{\lambda<\beta} X_\lambda)^{n(\omega)}$$

(this works for limit and succesor ordinals). So is clear that X_r is a subalgebra of A and contains G. □

[2]If you consider only infinite cardinals, it is enough to say 'least regular cardinal bigger than each $n(w)$'. Recall that r is *regular* iff is the supremum (sum) of strictly fewer, each strictly smaller, sets is (itself) strictly smaller. The first ones are $0, 1, 2, \aleph_0, \aleph_{\aleph_0}, \ldots$ Also, n is (capable of being) *cofinal* in $\aleph$ if there is a function $f : n \longrightarrow \aleph$ with $\forall \lambda \in \aleph \exists i \in n : f(i) \geq \lambda$

LECTURE 5

Yoneda Representation (9/23)

The natural operations on (Δ, n)-algebras have an algebraic structure. The derived operations on (Δ, n)-algebras (which are contained in the natural operations) also have an algebraic structure. How do this structures relate?

The answer will be easier introducing some more categorical concepts. Consider a ring Λ and the following 'equation':

$$\frac{\text{ring } \Lambda}{\mathrm{Hom}_\lambda(\Lambda, M) \cong M} : \frac{\text{categories}}{?} \tag{4}$$

What is '?'. In other words, what is the analogous in categories of the isomorphism between the set of ring-homomorphisms $\psi : \Lambda \longrightarrow M$ and M itself? (recall that this isomorphism is the correspondence $\psi \mapsto \psi(1)$).

The answer begins surprisingly; consider the functor

$$\mathcal{A}^{op} \times \mathcal{A} \xrightarrow{\mathcal{A}(-,-)} \mathcal{S}ets$$

defined as in the following diagram:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[d, "h"] \\ A' \arrow[u, "g"] & B' \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & \mathcal{A}(A,B) \arrow[dl, "- \circ g"'] \arrow[dr, "h \circ -)"] & \\ \mathcal{A}(A',B) \arrow[dr, "h \circ -"'] & & \mathcal{A}(A,B') \arrow[dl, "- \circ g"] \\ & \mathcal{A}(A',B') & \end{tikzcd}$$

Now recall the construction of the product functor in lecture 2:

$$\Theta : \mathcal{X} \times \mathcal{Y} \longrightarrow \mathcal{Z}$$

Let us see it in more detail: it defines two maps:

$$\mathcal{X} \xrightarrow{\widehat{\Theta}} \text{Func}\,(\mathcal{Y}, \mathcal{Z}) \qquad X \mapsto \Theta(X, -)$$

and

$$\mathcal{Y} \xrightarrow{\widetilde{\Theta}} \text{Func}\,(\mathcal{X}, \mathcal{Z}) \qquad Y \mapsto \Theta(-, Y)$$

From this point of view the map $\hom_{\mathcal{A}}$ can be interpreted

$$\widehat{\hom_{\mathcal{A}}} : \mathcal{A}^{op} \longrightarrow \text{Func}\,(\mathcal{A}, \mathcal{S}ets) \tag{5}$$

and

$$\widetilde{\hom_{\mathcal{A}}} : \mathcal{A} \longrightarrow \text{Func}\,(\mathcal{A}^{op}, \mathcal{S}ets) \tag{6}$$

It is a matter of taste which one we will study first. I would prefer to start with '$\widehat{\ }$' to avoid some confusion at first with the reversing of the arrows in the domain of '$\widetilde{\ }$'.

N. Yoneda (around the year 1954) was the first to call the attention to the marvelous properties of this functor. It is called now Yoneda representation, but it is more, in fact it is a full embedding.

Let us consider the diagram:

$$\mathcal{A} \xrightarrow{Y = \widetilde{\hom_{\mathcal{A}}}} \text{Func}(\mathcal{A}^{op}, \mathcal{S}ets) = \mathcal{S}ets^{\mathcal{A}^{op}}$$

$$\begin{tikzcd}
 & A \arrow[r] \arrow[dl, "\alpha"'] \arrow[dd, dashed, "f = \beta \circ \alpha"] & \mathcal{A}(-, A) \arrow[dd, "f \circ -"'] \arrow[dr, "\alpha \circ -"] & \\
B \arrow[dr, "\beta"'] & & & \mathcal{A}(-, B) \arrow[dl, "\beta \circ -"] \\
 & C \arrow[r] & \mathcal{A}(-, C) &
\end{tikzcd}$$

That is how $\widetilde{\hom}$ behaves: for each object $A \in \mathcal{A}$ you get a functor $\mathcal{A}(-, A) : \mathcal{A}^{op} \longrightarrow \mathcal{S}ets$ that behaves contravariantly (i.e. it 'reverses' the arrows), but much more: for each arrow f in $\mathcal{A}$ you get a natural transformation, $f \circ -$, between the corresponding (to the respective objects) functors.

Now we have a thorough description of a morphism

$$\mathcal{A}(A, C) \xrightarrow{Y} \text{nat}(\mathcal{A}(-, A), \mathcal{A}(-, C))$$

Yoneda's observations:

1. $\widetilde{\hom}$ *is 1-1 and onto.*
2. (really contained in (1)) Simplify notations for a possible proof and write $T = \mathcal{A}(-, C)$ so that

$$\mathcal{A}(A, C) \xrightarrow{Y} \mathrm{nat}(\mathcal{A}(-, A), \mathcal{A}(-, C))$$

becomes

$$T(A) \xrightarrow{Y} \mathrm{nat}(\mathcal{A}(-, A), T)$$

With this new presentation we have:

PROPOSITION 1 (Yoneda's Lemma). *For all* $T \in \mathcal{S}ets^{\mathcal{A}^{op}}$*, for all* $A \in |\mathcal{A}|$,

$$\mathrm{nat}(\mathcal{A}(-, A), T) \cong T(A)$$

This is the 'right' generality of the statements above. We could have proved (1) above, but in fact we will see that we do not need any general features of $\mathcal{A}(-, C)$ in the proof.

PROOF. Fix $\mathcal{A}$, fix $A \in |\mathcal{A}|$, fix $T : \mathcal{A}^{op} \longrightarrow \mathcal{S}ets$. First, let us contemplate for a while the following 'diagram':

$$T(A) \qquad\qquad\qquad \mathrm{nat}(\mathcal{A}(-, A), T)$$

Both sides are in splendid isolation ... How can you get from one to the other? Given $f \in TA$, you need to find a natural transformation $\varphi(f) : \mathcal{A}(-, A) \longrightarrow T$. A natural transformation is a family of arrows indexed by the objects in the category: So let us take $X \in |\mathcal{A}|$ (the common domain of both functors $\mathcal{A}(-, A)$ and T). Hence we are reduced to define the map $\{\varphi(f)\}_X : \mathcal{A}(X, A) \longrightarrow T(X)$. How can this be defined? You must take an arrow $a : X \longrightarrow A$ and get an element in $T(X)$. Here is the (straigthforward) solution:

$$\begin{array}{rcl} T(A) & \xrightarrow{\quad\varphi\quad} & \mathrm{nat}(\mathcal{A}(-, A), T) \\ f & \mapsto & \{\varphi(f)\}_X(a) = \{T(a)\}(f) \end{array} \tag{7}$$

Why φ is 1-1 and onto? Presumably from the existence of an inverse ψ for the map φ:

$$T(A) \xleftarrow{\ \psi\ } \mathrm{nat}(\mathcal{A}(-, A), T)$$

Recall the example of the ring in equation (4): take the identity to the identity. Let us do the 'same' here: if $\lambda \in \mathrm{nat}(\mathcal{A}(-, A), T)$, define

$$\psi(\lambda) = \lambda_A(\mathrm{id}_A) \tag{8}$$

Now come the verifications:

1. ψ is well defined

2. φ is well defined: is it really a natural transformation?
3. $\varphi \circ \psi = \mathrm{id}$
4. $\psi \circ \varphi = id$

All of them are relatively simple:

(1) is immediate.

(2)φ is a natural transformation. Fix $Y \xrightarrow{\xi} X$ in $\mathcal{A}$, and $f \in T(A)$.

(9)
$$\begin{tikzcd} \mathcal{A}(X,A) \arrow[r, "\varphi(f)_X"] \arrow[d, "-\circ\xi"'] & T(X) \arrow[d, "T(\xi)"] \\ \mathcal{A}(Y,A) \arrow[r, "\varphi(f)_Y"] & T(Y) \end{tikzcd}$$

To see that (9) commutes, let be $a \in \mathcal{A}(X,A)$, so we have:

$$\begin{tikzcd} a \arrow[r, dashed, "\varphi(f)_X"] \arrow[d, dashed, "-\circ\xi"'] & \varphi(f)_X(a) = \{T(a)\}(f) \arrow[d, dashed, "T(\xi)"] \\ a\circ\xi \arrow[r, dashed, "\varphi(f)_Y"] & \{T(a\circ\xi)\}(f) \overset{?}{=} T(\xi)[(T(a))(f)] \end{tikzcd}$$

T is a contravariant functor from the perspective of a, so the equality in the lower-right corner holds for all f.

(2) Why $\psi \circ \varphi = id$? Why, given a natural transformation $\lambda : \mathcal{A}(-,A) \longrightarrow T$, does $\{\varphi \circ \psi\} = \lambda$? We have to check

$$\forall X \in |\mathcal{A}|,\ [\{\varphi \circ \psi\}(\lambda)]_X = \lambda_X : \mathcal{A}(X,A) \longrightarrow T(X)$$

Now in order to check that these two functors are equal we have to evaluate them, that is: for all $a : X \longrightarrow A$ check

$$(\varphi(\psi(\lambda)))_X(a) = \lambda_X(a) \qquad (\in T(X)) \tag{10}$$

Now is time to start deciphering the meaning of (10). By definition of ψ:

$$(\varphi(\psi(\lambda)))_X(a) = (\varphi(\lambda_A(\mathrm{id}_A)))(a)$$

now applying definition of φ in (7)

$$= \{T(a)\}(\lambda_A(\mathrm{id}_A))$$

Finally, using the fact that λ is a natural transformation, is easy to see that the diagram on the left commutes, and hence the equality (10) follows (diagram on the right):

$$\begin{tikzcd} \mathcal{A}(A,A) \arrow[r, "\lambda_A"] \arrow[d, "- \circ a"'] & T(A) \arrow[d, "T(a)"] \\ \mathcal{A}(X,A) \arrow[r, "\lambda_X"] & T(X) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{id}_A \arrow[r, dashed, "\lambda_A"] \arrow[d, dashed, "- \circ a"] & \lambda_A(\mathrm{id}_A) \arrow[d, dashed, "T(a)"] \\ a \arrow[r, dashed, "\lambda_X"] & \lambda_X(a) = \{T(a)\}(\lambda_A(\mathrm{id}_A)) \end{tikzcd}$$

As you see, the proof goes in the only possible way you can go ...

(3) Why is $\psi \circ \varphi = \mathrm{id}_{T(A)}$? Let $f \in TA$, why is $\psi(\varphi(f)) = f$? Applying the definitions of ψ and φ we have:

$$\begin{aligned} \psi(\varphi(f)) &= \varphi(f)_A(\mathrm{id}_A) \\ &= T(\mathrm{id}_A)(f) \\ &= \{\mathrm{id}_{TA}\}(f) = f \end{aligned}$$

□

All this is exactly was is going in the case of rings (recall 'equation' (4)) with the isomorphism

$$\hom_\Lambda(\Lambda, M) \xrightarrow{\cong} M \qquad \lambda \mapsto \lambda m$$

The only thing is that in Λ there is only one element, but in *Cat* there are many objects which plays the role of 1 in the original construction (see equation (8)).

LECTURE 6

More on constructions (9/25)

We could summarize Yoneda Lemma saying that there is an explicit map between the upside and the lowerside paths in the following diagram[1]:

$$\begin{tikzcd}
 & (\mathcal{S}ets^{\mathcal{A}^{op}})^{op} \times \mathcal{S}ets^{\mathcal{A}^{op}} \arrow[dr, "\mathrm{hom}"] & \\
\mathcal{A}^{op} \times \mathcal{S}ets^{\mathcal{A}^{op}} \arrow[ur, "Y^{op} \times \mathrm{id}"] \arrow[dr, "evaluation"'] & & SET \\
 & \mathcal{S}ets \arrow[ur] &
\end{tikzcd}$$

That is, for every $A \in \mathcal{A}^{op}$ and $T \in \mathcal{S}ets^{\mathcal{A}^{op}}$, the functors $\mathrm{nat}(\mathcal{A}(-, A), T)$ and $T(A)$ are *isomorphic.* Nice presentation if we discount the fact that we do not know what 'isomorphic' is ... but:

DEFINITION 4. Let $\mathcal{X}$ be a category. Two objects $A, B \in \mathcal{X}$ are *isomorphic* (in $\mathcal{X}$) if there are $\mathcal{X}$-morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ such that:

1. $g \circ f = \mathrm{id}_A$
2. $f \circ g = \mathrm{id}_B$

We proved only that $T(A) \xrightarrow{Y} \mathrm{nat}(\mathcal{A}(-, A), T)$ is a bijection. But there is more:

PROPOSITION 2.

$$\varphi = \varphi_{(A,T)} : T(A) \longrightarrow \mathrm{nat}(\mathcal{A}(-, A), T) \tag{11}$$

is natural *in* $A \in |\mathcal{A}^{op}|$ *and* $T \in |\mathcal{S}ets^{\mathcal{A}^{op}}|$

[1]Two remarks need to be made: (1) Given a functor $F : \mathcal{X} \longrightarrow \mathcal{Y}$, it is straightforward to check that $F^{op} : \mathcal{X}^{op} \longrightarrow \mathcal{Y}^{op}$ is a functor. (2) SET could be 'enourmous'.

PROOF. Exercise: use heavily the fact that T is a functor. $\square$

The following corollary is used in practive more than Yoneda Lemma itself.

COROLLARY 1. *If for objects $A, B \in |\mathcal{A}|$, $\mathcal{A}(-,A)$ and $\mathcal{A}(-,B)$ are isomorphic in $\mathcal{S}ets^{\mathcal{A}^{op}}$, then A and B are isomorphic in $\mathcal{A}$.*

PROOF. Consider

$$\begin{array}{ccc} \mathcal{A}(A,B) & \xrightarrow[\text{1-1, onto}]{Y} & \mathrm{nat}(\mathcal{A}(-,A),\mathcal{A}(-,B)) \\ A \xrightarrow{f} B & & \xrightarrow{Y(f)} \\ A \xleftarrow{g} B & & \xleftarrow{Y(g)} \end{array}$$

The facts that $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$ in the suitable places plus the naturality in (11) above gives us the desired result. $\square$

How can this corollary help us in showing that two objects are isomorphic? Let us see some examples.

Take $T \in |\mathcal{S}ets^{\mathcal{A}^{op}}|$. Think the easiest functor you can imagine: $T(A) = \mathbf{1}$. This functor is covariant and contravariant:

$$\mathcal{A}(X,*) \cong T(X) \cong \mathbf{1}$$

$\mathcal{S}ets^{op}$ is a sort of category of "wish-lists" for desirable objects $\mathcal{A}$ should have. For example, in differential equations, when looking for an analytic solution of a certain equation, you first obtain a generalized function solution (a distribution) and then you particularize and try to get the right one with the desired properties. In general, you look for the general solution to a problem in a place where it is easier to find, and *then* look down if you have an isomorphic object in the right place.

In the same way as the Yoneda functor is called Yoneda representation, i.e. represents the objects of A as the functor $\mathcal{A}(-,A)$.

Terminal objects.

DEFINITION 5. If the functor

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T} & \mathcal{S}ets \\ A & \mapsto & \mathbf{1} \\ \xrightarrow{\alpha} & \mapsto & \mathrm{id}_1 \end{array}$$

is representable, say $T(A) \cong \mathcal{A}(-,*)$ for some $* \in |\mathcal{A}|$, then $*$ is called a *terminal object* of $\mathcal{A}$.

EXAMPLES.

1. In (abelian) groups: $\{0\}$ is a terminal object.
2. In **Top**, the one-point space is a terminal object.

3. In rings with unit: depend on what do you mean by ring with unit.
   (a) If you allow $1 = 0$ then the one-element ring is a terminal object.
   (b) If you require $1 \neq 0$, there is no terminal object.
4. Δ-algebras: the one-point Δ-algebras are terminal objects.

As you can see, in most of concrete cases, terminal objects exist, and looks like one-element objects.

(5) Now consider the category $\mathcal{C}_{(X,\leq)}$. An element $* \in \mathcal{C}$ is terminal in $\mathcal{C}$ if and only if $*$ is a top-element in the order of X. So, for example, $\mathcal{C}_{(\mathbb{N},\leq)}$ has no terminal object, while $\mathcal{C}_{(\mathbb{N},\geq)}$ has a terminal object.

(6) (one-minute quiz) What does it mean for $\mathcal{M}_M$ to have a terminal object?

It is easy to see that the functor $T : \mathcal{A}^{op} \longrightarrow \mathcal{S}ets$, with $T(A) = \mathbf{1}$ is terminal in the category $\mathcal{S}ets^{\mathcal{A}^{op}}$:

$$\begin{tikzcd} A & F(A) \arrow[d, "F(\alpha)"] \arrow[dr] & \\ & & T(A) = T(B) \\ B \arrow[uu, "\alpha"] & F(B) \arrow[ur] & \end{tikzcd}$$

is a natural transformation by default. This construction is so general that it applies to almost every case (of course there are exceptions).

Products. The way to obtain interesting properties in a category is to expand interesting construction in $\mathcal{S}ets$ to $\mathcal{S}ets^{\mathcal{A}^{op}}$. Given two functors $S, T \in \mathcal{S}ets^{\mathcal{A}^{op}}$,

$$\begin{tikzcd} P(A) & =S(A) \times T(A) \\ P(B) \arrow[u] & =S(B) \times T(B) \arrow[u, "S(\alpha) \times T(\alpha)"'] \end{tikzcd}$$

If $S = Y(X)$ and $T = Y(Z)$, what can $P = S \times T$ be representing? Let us see; we have:

$$\mathcal{A}(-, ?) \cong \mathcal{A}(-, X) \times \mathcal{A}(-, Z)$$

now considering $- =?$ we are the following situation:

$$\begin{tikzcd}
& - \arrow[d] \arrow[dl] \arrow[dr] & \\
X & ? \arrow[l] \arrow[r] & Z
\end{tikzcd}$$

Now think in topological groups, vector spaces, Δ-algebras, etc. What is a solution to this diagram? In the case of Posets the answer is ilustrative

$$\begin{tikzcd}
& & x \\
- \arrow[r, "\leq"] \arrow[urr, "\leq"] \arrow[drr, "\leq"'] & p \arrow[ur, "\leq"'] \arrow[dr, "\leq"] & \\
& & z
\end{tikzcd}$$

Here the "product" p is the glb(x, z) (or $\inf(x, z)$)(or $x \wedge z$).

In the majority of the set-based categories the product means this. But the product is not generally available: Take for example the category of fiels. It is easy to realize that there can be no product there.

As we saw, the product object arises as the product of the representation. We can make this construction more general: nothing prevent us for taking more than two elements. So, let I an index family, and take a map $I \xrightarrow{X} \mathcal{A}$, that is a family of objects $(X_i)_{i\in I} \subseteq |\mathcal{A}|$.

$$\begin{tikzcd}
& & - \arrow[d, "t"] \arrow[ddll, "t_j"'] \arrow[ddrr] & & \\
\cdots & & \prod\limits_{i\in I} X_i \arrow[dll, "\pi_j"'] \arrow[drr] & & \cdots \\
\cdots X_j & & \cdots & & \cdots
\end{tikzcd}$$

such that $(\cdots, t_j = \pi_j \circ t, \cdots)_{j\in I}$.

There may be no such object in $\mathcal{A}$. Though it is clear that looking at $Y(X_j) \in \mathcal{S}ets^{\mathcal{A}^{op}}$ there *will* be

$$\prod_{i\in I} Y(X_i)$$

This can be sumarized asking: there exists $\prod_{i\in I} X_i$ such that

$$Y(\prod_{i\in I} X_i) \cong \prod_{i\in I} Y(X_i)$$

(In words: Does Yoneda recognizes the proposed object as the product?)

Equalizers. A small graph in general is *not* a category (the composition of arrows is missed), but there are some cases in which it is:

$$
\begin{array}{ll}
(a) & \bullet \\
(b) & \bullet \rightrightarrows \bullet \\
(c) & \begin{tikzcd} \bullet \arrow[dr] & \\ & \bullet \\ \bullet \arrow[ur] & \end{tikzcd}
\end{array}
$$

We will see they play an important rol in categorical concepts. Take now, for example, the known construction in graph theory:

$$\{g \in G : s(g) = r(g)\} \subseteq G \overset{r}{\underset{s}{\rightrightarrows}} H$$

What is this set? It is just the *equalizer* (also called *difference-kernel*) of s and r (the collections of witnesses that think -by personal experience!- that s and r are equal, no matter what s or r do to their neighbors).

DEFINITION 6. Given two maps in $\mathcal{A}$ with common source and target

(12) $$E \xrightarrow{e} A \overset{\alpha}{\underset{\beta}{\rightrightarrows}} B$$

an *equalizer* for (α, β) is a map e as in (12) such that in the following diagram

$$
\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & B \\
- \arrow[u, dotted, "\exists!\, \tilde{t}"] \arrow[ur, dotted, "\forall t"'] & &
\end{tikzcd}
$$

for all arrow t such that $\alpha\, t = \beta\, t$, then there is a unique arrow $\tilde{t}$ that factors through e, i.e. $t = e\,\tilde{t}$

What is really happening here is:

$$Y(E) \xrightarrow{\;\;e\;\;} Y(A) \underset{Y(\beta)}{\overset{Y(\alpha)}{\rightrightarrows}} Y(B)$$

$$\mathrm{eq}(-, \begin{smallmatrix}\alpha\circ- \\ \beta\circ-\end{smallmatrix}) \subseteq \mathcal{A}(-,A) \underset{\beta\circ-}{\overset{\alpha\circ-}{\rightrightarrows}} \mathcal{A}(-,B)$$

This are two point of viewing the same thing. Now, when you have products and equalizers you have a lot more.

LECTURE 7

Free Δ-algebras (9/30)

We are going into some important concepts in (Δ, n)-algebras, namely "equations", "equationally definable", an so on.

First, it is good to recall here some aspects of the Yoneda representation we saw in lecture 5. If in the Yoneda map (see equation (6) there) we put $\mathcal{A} = \mathcal{B}^{op}$ (and hence $\mathcal{B} = \mathcal{A}^{op}$), we get the following map (see also equation (5)):

$$\begin{array}{ccc} \mathcal{B}^{op} & \xrightarrow{Y} & \mathcal{S}ets^{\mathcal{B}} \\ A & \mapsto & \mathcal{B}(A, -) \end{array}$$

that is

$$\mathcal{B}(A, C) = \mathcal{B}^{op}(C, A) \cong \mathrm{nat}(\mathcal{B}(A, -), \mathcal{B}(C, -))$$

and we can easily check that the covariant functor have no more (nor less) natural transformations than the contravariant one in (6). More generally we have (the arguments are the same as in lecture 5):

$$T(A) \cong \mathrm{nat}(\mathcal{B}(A, -), T)$$

which is nothing else than the other version of the Yoneda map.

With this observations, we can prove the following important result:

THEOREM 1. *Given a set Δ and "arity function n", and given a set[1] k, it turns out that the covariant set-valued functor*

$$(\Delta, n)\text{-}\mathcal{A}lg \xrightarrow{(U_\Delta)^k} \mathcal{S}ets$$

is representable, indeed we have

$$(U_\Delta)^k \cong Y(F(k)) \cong Y(P_k)$$

where $F(k)$ and P_k will be defined below.

[1] Again, we are using the letter k to suggest psicologically a natural number, but could be any set.

Let us explain the contents of this proposition before proving it. We are asserting that there is $F(k) \in |(\Delta, n)\text{-}\mathcal{A}lg|$ for which $(U_\Delta)^k \cong Y(F(k))$. That is for each (Δ, n)-algebra A

$$\mathcal{S}ets(k, U_\Delta(A)) \cong U^k(A) \cong Y(F(k))(A) \cong \Delta\text{-}\mathcal{A}lg(F(k), A)$$

Think in more familiar examples, like groups, rings, vector spaces:

1. Groups: On the left, the elements are functions $i \mapsto g_i$, which defines a free group in k generators $FG[\cdots g_i \cdots]$ that is, words in g_i and $\bar{g}_i$ (f. ex. $g_i g_j \bar{g}_i$ should be thought as $x_i x_j x_i^{-1}$) identifying $g_i \bar{g}_i$, e and $\bar{g}_i g_i$. (formally: $g_i \bar{g}_i \sim e \sim \bar{g}_i g_i$). On the right:

$$\begin{tikzcd} k \arrow[r, "i \mapsto x_i"] \arrow[d, "i \mapsto g_i"'] & G \\ FG[\cdots g_i \cdots] \arrow[ur, "g_i \mapsto x_i,\ \bar{g}_i \mapsto x_i^{-1}"'] & \end{tikzcd}$$

2. Rings: On the left, the map $i \mapsto g_i$ give us the ring $\mathbb{Z}[\cdots g_i \cdots]_{i \in k}$. On the right hand side, we have the ring of polynomials in k free generators (Think of the elements of $F(k)$, the polynomials, as 'operators' from $A^m \longrightarrow A$ defined as the evaluation in each variable).

$$\begin{tikzcd} k \arrow[r, "i \mapsto r_i"] \arrow[d, "i \mapsto g_i"'] & R \\ \mathbb{Z}[\cdots g_i \cdots] \arrow[ur, "g_i \mapsto r_i \quad \text{(extend `naturally')}"'] & \end{tikzcd}$$

3. Vector spaces: On the left hand side, we have $\mathbb{R}^{(k)}$. On the right, we find linear transformations extended by linearity.

$$\begin{tikzcd} k \arrow[r, "i \mapsto v_i"] \arrow[d, "i \mapsto g_i"'] & V \\ \mathbb{R}^{(k)} \arrow[ur, "g_i \mapsto v_i \quad \text{(extend lineally)}"'] & \end{tikzcd}$$

Once one of this examples rang the bell in your brain, we can continue with the general case, that is the proof of the proposition 1.

PROOF. [of proposition 1] Let

$$\mathcal{F}_k = \mathrm{nat}((U_\Delta)^k, U_\Delta)$$

endowed with structure of a (possible too large) Δ-algebra. Let F_k the subalgebra of $\mathcal{F}_k$ generated by the k^{th} projection transformation.

Now let us define P_k. We will define it inductively:

$$P_k = \bigcup_{\alpha < r} X_\alpha = X_r$$

where r is defined as in the last section of lecture 4, and X_α is

$$X_\alpha = k \cup \bigcup_{\omega\in\Delta}\left(\{w\}\times(\bigcup_{\beta<\alpha} X_\beta)^{n(\omega)}\right)$$

So F_k is a set even though $\mathcal{F}_k$ could be a big class. (Notice the 'similarity' with the construction in lecture 4: there we wrote $\omega(a)$, here we have $\{\omega\}\times a$). In P_k the elements are tuples. (Think for a moment in rings again: take $f_1,\ldots,f_n$ polynomials in the variables $\bar{x}=(x_1,\ldots,x_t)$, p a polynomial in n variables, then $p(f_1,\ldots,f_n)$ *is* a polynomial in $\bar{x}$, namely $p(f_1(\bar{x}),\ldots,f_n(\bar{x}))$.

$$\begin{tikzcd}
k = X_0 \subseteq \cdots \subseteq X_\beta \subseteq \cdots \subseteq X_\alpha \subseteq \cdots \subseteq \arrow[d, "a"'] & \arrow[r, Rightarrow, "\text{restriction to } k"] & \arrow[l, Leftarrow, "?"] & P_k \arrow[d, "\Delta\text{-homo}"] \\
U_\Delta(A) & & & A
\end{tikzcd}$$

The construction in $\Rightarrow$-direction is done by induction. We are going to illustrate it by an example: $X_0 \subseteq X_1$. We know

$$X_1 = X_0 \cup \bigcup_{\omega\in\Delta}\left(\{\omega\}\times X_0^{n(\omega)}\right)$$

In X_0 the map is already given: $a : X_0 \longrightarrow |A|$ (recall the notation $|A| \equiv U_\Delta(A)$). So for the rest, let us take $\omega\in\Delta$,

$$\Leftarrow : X_0^{n(\omega)} \xrightarrow{a} |A|^{n(\omega)} \xrightarrow{\omega} |A|$$

It is clear that this construction obeys $(\Rightarrow \circ \Leftarrow) = \text{id}$ and $(\Leftarrow \circ \Rightarrow) = \text{id}$ and $\Rightarrow$ and $\Leftarrow$ are 1-1 and onto.

P_k is called, in universal algebra, *functionally (totally) 'free* Δ-algebra, or also *polynomially free* Δ-algebra.

Now let us go to the second part of the proof:

$$(U_\Delta)^k \cong Y(F(k))$$

Consider the diagram (j is the inclusion):

$$\begin{tikzcd}
k \arrow[rr] \arrow[dr] \arrow[d, "a"'] & & \mathcal{F}_k \arrow[d] \\
{|A|} & F_k \arrow[ur, "j"'] \arrow[d] & {|A|} \\
& A &
\end{tikzcd}$$

The tricky part is given $k \xrightarrow{a} |A|$ to get a map $F_k \longrightarrow A$. What we will do is to construct an arrow $\bar{a} : \mathcal{F}_k \longrightarrow A$ such that $\bar{a}|_k = a$, so the restriction of $\bar{a}$ to F_k will give us the desired arrow.

How can we expect there is such a thing? Consider

$$\lambda : U_\Delta^k \longrightarrow U_\Delta$$
$$\lambda_A : (U_\Delta(A))^k \longrightarrow U_\Delta(A)$$

What is the only piece of information we have?: the arrow a. So $\lambda_A(a) \in |A|$. Is $\lambda \mapsto \lambda_A(a)$ a Δ-homomorphism from $\mathcal{F}_k \longrightarrow A$? Let $\omega \in \Delta$, let $\xi \in |\mathcal{F}_k|^{n(\omega)}$. For each $i \in n(\omega)$, we have $\xi_i \in \mathcal{F}_k$, so we can apply $(\xi_i)_A(a) \in A$. Thus

$$\begin{tikzcd}
(\mathcal{F}_k)^{n(\omega)} \arrow[r] \arrow[d, "\omega"'] & {|A|^{n(\omega)}} \arrow[d, "\omega"] \\
\mathcal{F}_k \arrow[r] & {|A|}
\end{tikzcd}$$

so taking $\xi \in (\mathcal{F}_k)^{n(\omega)}$ we have:

$$\begin{tikzcd}
\xi \arrow[rr, dashed] \arrow[d, dashed, "\omega"'] & & (\cdots(\xi_i)(a)\cdots)_{i\in n(\omega)} \arrow[d, dashed, "\omega"] \\
\omega(\cdots\xi_i\cdots) \arrow[r, dashed] & (\omega(\cdots\xi_i\cdots))_A(a) \overset{?}{=} & \omega(\cdots(\xi_i)_A(a)\cdots)
\end{tikzcd} \tag{13}$$

Why this are the same? You have just to recall the definitions of ω, $\mathcal{F}_k$, etc.:

$$\xi_i : U_\Delta^k \longrightarrow U_\Delta \qquad (i \in n(\omega))$$

also recall

$$\omega(\cdots\xi_i\cdots) = U_\Delta^k \xrightarrow{(\cdots\xi_i\cdots)_{i\in n(\omega)}} U_\Delta^{n(\omega)} \xrightarrow{\omega} U_\Delta$$

So in the attempt to evaluate the path through the left-bottom corner in the diagram (13) we get the path through the upper-right corner. So finally consider the map $a \mapsto \bar{a}|_{F_k}$ and we are done. □

So $Y(F_k)$ and $Y(P_k)$ are isomorphic to a third thing, ergo

$$Y(F_k) \cong Y(P_k)$$

Also we have for $\mathcal{F}_k$ the following:

$$\mathcal{F}_k \cong \mathrm{nat}(U_\Delta^k, U_\Delta) \tag{14}$$

$$\cong \mathrm{nat}(\mathcal{A}(F_k, -), U_\Delta) \tag{15}$$

$$\cong U_\Delta(F_k) \tag{16}$$

The step from (14) to (15) is because of the observation we made at the beginning of the lecture, and from (15) to (16) is just Yoneda Lemma.

An important case is the particular case $k = 1$

$$\mathrm{nat}(U_\Delta, U_\Delta) \cong \mathcal{F}_1$$

which are the derived unary operations.

LECTURE 8

Equationally defined classes (10/2)

EQUATIONS. Given (Δ, n) (some people have called this *species*) and considering (Δ, n)-algebras and the functor

$$U_{(\Delta,n)} : (\Delta, n)\text{-}\mathcal{A}lg \longrightarrow \mathcal{S}ets$$

by a (Δ, n)-*equation* in "set-of-variables" X, mean any ordered pair $e = (e_1, e_2)$ of natural transformations

$$U^X_{(\Delta,n)} \underset{e_2}{\overset{e_1}{\rightrightarrows}} U_{(\Delta,n)} \tag{17}$$

Equivalently[1], any pair of members of $U_{(\Delta,n)}(F_{(\Delta,n)}(X))$.

VALIDITY. If $(A, \{A_\omega\}_{\omega\in\Delta})$ is a (Δ, n)-algebra, we say e is *valid* (holds) in $(A, \{A_\omega\}_{\omega\in\Delta})$ if

$$(e_1)_{(A,(A_\omega)_\omega)} = (e_2)_{(A,(A_\omega)_\omega)}$$

Generally, one asks about an entire class $\mathcal{E}$ of equations, whether in a given algebra, *every* equation from $\mathcal{E}$ is valid. So let us denote

$$((\Delta, n), \mathcal{E})\text{-}\mathcal{A}lg = \{A \in |\Delta\text{-}\mathcal{A}lg| \ : \ \text{each eq. of } \mathcal{E} \text{ holds in } A\}$$

It is immediate that

$$((\Delta, n), \mathcal{E})\text{-}\mathcal{A}lg \subseteq (\Delta, n)\text{-}\mathcal{A}lg$$

VARIETIES. Any category of the form $((\Delta, n), \mathcal{E})$-$\mathcal{A}lg$ for some set Δ, arity function n and class $\mathcal{E}$ is a *variety* (in the Universal Algebraist "lingo"), or an *equationally definable class* of $((\Delta, n), \mathcal{E})$-algebras.

Group theorists like to think in terms of the signature $\{\circ, (\ )^{-1}, e\}$. Then they restrict to study "subvarieties" (abelian groups, p-groups, etc.). Lattice theorists think in term of the signature $(\vee, \wedge, 1, 0)$. Also they are interested in particular varieties: distributive, modular lattices, etc. Then they go to sub-sub-varieties an so on.

[1] In eq. (17)'s setting, what you ask is what you get. In $U_{(\Delta,n)}(F_{(\Delta,n)}(X))$ things are somewhat encoded.

QUESTION. One is tempted to try to characterize the general class

(18) $\quad |\mathcal{X}| \subseteq ((\Delta, n), \mathcal{E})\text{-}\mathcal{A}lg \qquad$ Birkhoff's Thm.

or in a different setting

$$\begin{tikzcd} \mathcal{X} \arrow[d, "U"'] \\ \mathcal{S}ets \end{tikzcd} \qquad \text{Beck's Thm.}$$

What can you say of U or $\subseteq$ that would be a statement neccesary and sufficient for the class $\mathcal{X}$ to be a variety?

Historically first Birkhoff's theorem provided an answer in the language of HSP-classes. The other question was answered by Jon Beck in the early sixties. We will see that Beck's theorem provides a simpler and cleaner way of stating the problem above. In order to formulate Birkhoff's theorem we need to define before what are subalgebras, quotient algebras, congruence algebras and HSP classes. Let us proceed in this order.

SUBALGEBRAS. Fix $(\Delta, n), \mathcal{E}$ and let $\mathcal{V} = ((\Delta, n), \mathcal{E})\text{-}\mathcal{A}lg$. Let A be an algebra from $\mathcal{V}$. What does it take for $X \subseteq |A|$ to be an element of $\mathcal{V}$?

It is worthwhile to see in parallel both pictures to see what is going on in each level: $\Delta\text{-}\mathcal{A}lg$ and $\mathcal{S}ets$.

$$\begin{tikzcd} \mathcal{V} & X \arrow[d, dashed] & A \arrow[d, dashed] \\ \mathcal{S}ets & X \subseteq & {|A|} \end{tikzcd}$$

Let state first an elementary observation:

PROPOSITION 3. *The following statements are equivalent:*

1. *A subset $X \subseteq |A|$ "is" (the underlying set of) an element of $\mathcal{V}$, and the inclusion $X \xrightarrow{j} |A|$ the underlying functor for a homomorphism $X \longrightarrow A$ of $\mathcal{V}$.*
2. *For all $\omega \in \Delta$, for all $n(\omega) \xrightarrow{x} X$, the effect $\omega(j \circ x) \in X$ $(\subseteq |A|)$ i.e. subset X is closed in $|A|$ under all the operations ω from Δ.*

PROOF. (1) $\Rightarrow$ (2): j is the underlying functor $j = |f|$ of a homomorphism $f : B \longrightarrow A$, then inspect

$$\begin{tikzcd} X^{n(\omega)} = |B|^{n(\omega)} \arrow[r, "|f|^{n(\omega)}", "j^{n(\omega)}"'] \arrow[d, "\omega"'] & |A|^{n(\omega)} \arrow[d, "\omega"] \\ X = \quad |B| \arrow[r, "|f|", "j"'] & |A| \end{tikzcd}$$

to see that the desired conclusion is valid.

Conversely ((2) $\Rightarrow$ (1)): Suppose (2) happens. The equation $\omega(j \circ x) = j(?)$ has unique solution, so it is $? = \omega(x)$. (in fact this is just to say: declare the operation in the subspace to be the same as in the bigger space, because it does what I want it to do). So making a (Δ, n)-algebra out of X such that j "becomes" a homomorphism is easy.

But, why do equations from $\mathcal{E}$ hold there? Let's see: If $B \xrightarrow{f} C$ are two Δ-algebras and f is a homomorphism, and if $|f|$ is 1-1, then, whenever (e_1, e_2) is an equation C satisfies, then B will satisfy it too! see the diagram:

$$\begin{tikzcd} |B|^k \arrow[r, "|f|^k"] \arrow[d, shift right, "(e_1)_B"'] \arrow[d, shift left, "(e_2)_B"] & |C|^k \arrow[d, shift right, "(e_1)_C"'] \arrow[d, shift left, "(e_2)_C"] \\ |B| \arrow[r, "1-1", "|f|"'] & |C| \end{tikzcd}$$

Whichever way I commute the switch (Whichever path I choose at the fork) I get the commutative square. So we have

$$|f| \circ (e_1)_B = |f| \circ (e_2)_B$$

Now applying this maps to $b \in |B|$ and using that $|f|$ is 1-1 we can easily check that $(e_1)_B = (e_2)_B$. $\square$

$|\mathcal{X}|$ is an *S-class* in (Δ, n)-$\mathcal{A}lg$ if it is "closed under the formation of subalgebras". For example, abelian groups is a S-subclass of groups.

QUOTIENT ALGEBRAS. If $B \xrightarrow{f} C$ are two Δ-algebras and f is a homomorphism, and if $|f|$ is onto, then, whenever (e_1, e_2) is an equation

C satisfies, then B will satisfy it too! The proof is basically the same as before:

$$\begin{tikzcd}
|B|^k \arrow[r, "|f|^k"] \arrow[d, shift right, "(e_1)_B"'] \arrow[d, shift left, "(e_2)_B"] & |C|^k \arrow[d, shift right, "(e_1)_C"'] \arrow[d, shift left, "(e_2)_C"] \\
|B| \arrow[r, "\text{onto}", "|f|"'] & |C|
\end{tikzcd}$$

If $|f|$ is onto, then $|f|^k$ is onto (by the Axiom of Choice), so

$$(e_1)_C \circ |f|^k = |f| \circ (e_1)_B = |f| \circ (e_2)_B = (e_2)_C \circ |f|^k$$

so now apply to σ_k in both sides. (The same law of "cancellation" $|f|^k \circ \sigma_k = \text{id}$ that holds in the previous case, holds here in the 'other' side).

So we can ask here: Given B a $((\Delta, n), \mathcal{E})$-algebra, X a set, and $f : |B| \longrightarrow X$ onto in $\mathcal{S}ets$, when there exists a $((\Delta, n), \mathcal{E})$-algebra C such that $U_\Delta(C) = X$ and f homomorphism?

$$\begin{tikzcd}
\mathcal{V} & B \arrow[r, "f"] \arrow[d, dashed] & C \arrow[d, dashed, "\text{when?}"] \\
\mathcal{S}ets & {|B|} \arrow[r, "\text{onto}", "|f|"'] & X\ (= |C|)
\end{tikzcd}$$

PROPOSITION 4. *The following statements are equivalent:*

1. *There exists a unique algebra substructure C "on X" ($|C| = X$) such that $p = |f|$ for some (unique) $f : B \longrightarrow C$.*
2. *$X =_{def} \{(b_1, b_2) \in |B| \times |B| : p(b_1) = p(b_2)\}$ is a subalgebra of $B \times B$.*

DISGRESSION. If X and Y are two (Δ, n)-algebras, we may take $|X| \times |Y|$ and try to impose a (Δ, n)-algebra structure on it. Optimistically hoping that the projection functions

$$\begin{tikzcd}
 & {|X| \times |Y|} \arrow[dl] \arrow[dr] & \\
{|X|} & & {|Y|}
\end{tikzcd}$$

turns out to be homomorphisms of (Δ, n)-algebras, i.e. given $\omega \in \Delta$ (arity $n(\omega)$) and a tuple $n(\omega) \xrightarrow{z} |X| \times |Y|$, seek $\omega(z) \in |X| \times |Y|$ such that $\omega(z)_X = \omega(z_X)$ and $\omega(z)_Y = \omega(z_Y)$, i.e.

$$\begin{tikzcd}
 & {(|X| \times |Y|)^{n(\omega)}} \arrow[dl] \arrow[d, "\omega"] \arrow[dr] & \\
{|X|^\omega} \arrow[d, "\omega"'] & {|X| \times |Y|} \arrow[dl, "\mathrm{pr}_X"] \arrow[dr, "\mathrm{pr}_Y"] & {|Y|^\omega} \arrow[d, "\omega"] \\
{|X|} & & {|Y|}
\end{tikzcd}$$

The specification of the nature of the problem is the solution ot it!:

$$\begin{aligned}
z &= (\cdots z_i \cdots) \\
&= (\cdots (x_i, y_i) \cdots) \\
\text{so } \omega(z) &= (\omega(x_i)_{|f|}, \omega(y_i)_{|f|})
\end{aligned}$$

A family of maps $\{f_\alpha\}$ is jointly 1-1 if

$$\forall x_1, x_2 [\forall \alpha f_\alpha(x_1) = f_\alpha(x_1)] \Rightarrow x_1 = x_2$$

Note that projections are jointly 1-1.

Next class: the construction of p.

LECTURE 9

Congruence relations and quotient algebras (10/7)

Congruence relations. Let us review congruence relations in the category $\mathcal{S}ets$. Given sets A, B and a function $f : A \longrightarrow B$, a *congruence relation* of f is the set

$$\equiv_f = \{(x_1, x_2) \in A \times A : f(x_1) = f(x_2)\} \subseteq A \times A$$

It follows immediately that it is (as a subset of $A \times A$, as a binary relation on A) a reflexive, symmetric and transitive (RST) relation on A.

Conversely, given a RST relation E on A, there is a map $f : A \longrightarrow ?$ with $E = \equiv_f$ (Just define $f : A \longrightarrow P(A)$ with $f(a) = \{x \in A : (a, x) \in E\}$. cf. Halmos).

Let us phrase the above concepts in general terms. Let us start with the concept of congruence relation:

KERNEL PAIRS. First we have two maps, x_1, x_2, from $\equiv_f$ to A

$$\begin{tikzcd} \equiv_f \arrow[r] \arrow[rr, dashed, bend left, "x_1"] \arrow[rr, dashed, bend right, "x_2"'] & A \times A \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Notice that x_1, x_2 are jointly monic. Thus, we have a diagram

(19)
$$\begin{tikzcd} \equiv_f \arrow[rr, shift left, "x_1"] \arrow[rr, shift right, "x_2"'] & & A \arrow[r, "f"] & B \\ & T \arrow[ul, "\exists! a"] \arrow[ur, shift left, "a_1"] \arrow[ur, shift right, "a_2"'] & & \end{tikzcd}$$

with $f \circ x_1 = f \circ x_2$ and whenever T and a_1, a_2 satisfy $f \circ a_1 = f \circ a_2$, there is a unique $a : T \longrightarrow \equiv_f$ such that $a_\alpha = x_\alpha \circ a$ $(\alpha = 1, 2)$. We say that x_1, x_2 is a *kernel pair* for f.

We will see that we can also state the concepts 'jointly monic', 'reflexivity', 'symmetry' and 'transitivity' (JM,R,S,T) in categorical terms, and so state the above result in terms that are totally general:

PROPOSITION 5 (in $\mathcal{S}ets$). [1] *A pair of maps $E \rightrightarrows A$ is a kernel pair of some $f : A \longrightarrow ?$ if and only if*

$$(E \rightrightarrows A) : E \longrightarrow A \times A$$

is JMRST (jointly-monic, reflexive, symmetric, transitive).

So let us state what does it mean to be jointly monic, reflexive, summetric and transitive relations in the general context. Let us try to generalize these concepts to a general category. So assume $\mathcal{A}$ is an arbitrary category, A, B objects and $A \xrightarrow{f} B$ and arrow in $\mathcal{A}$.

JOINTLY MONIC. The arrows x_1, x_2 are *jointly monic* if whenever we have the following diagram

$$\begin{tikzcd} T \arrow[r, shift left, "t_1"] \arrow[r, shift right, "t_2"'] & E \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & A \end{tikzcd}$$

with $x_1t_1 = x_1t_2$ and $x_2t_1 = x_2t_2$, then $t_1 = t_2$.

It is clear that a kernel pair is jointly monic: If $x_1t_1 = x_1t_2$ and $x_2t_1 = x_2t_2$ as above, why does $t_1 = t_2$? Because (by diagram (19)) there is a unique $t : T \longrightarrow E$ with $x_1t = x_1t_1$ and $x_2t = x_2t_2$and hence $t_1 = t = t_2$.

REFLEXIVITY. Look at the diagram

$$\begin{tikzcd} E \arrow[r, shift left=2, "x_1"] \arrow[r, shift right=2, "x_2"'] & A \arrow[l, dashed, "\Delta"'] \\ & A \arrow[ul, "\exists!\Delta"] \arrow[u, shift left, "\mathrm{id}"] \arrow[u, shift right, "\mathrm{id}"'] \end{tikzcd}$$

That is what reflexivity is: the existence of a map Δ back form A to E, such that $\forall i = 1, 2 : x_i \circ \tau = x_{3-i}$.

Again, it is easy to see that a kernel pair is reflexive.

SYMMETRY. This should mean that the process of flipping elements in A does not matter. We just have to get a free-element way of describing

[1]The 'if' part of this proposition does no hold in an arbitrary category (one reason: kernel pairs could not exist).

that flip.

$$
\begin{tikzcd}
E \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & A \\
& E \arrow[u, shift right, "x_2"] \arrow[u, shift left, "x_1"'] \arrow[ul, "\exists!\tau"]
\end{tikzcd}
$$

with $x_1 \circ \tau = x_2$ and $x_2 \circ \tau = x_1$.

Notice again that a kernel pair is symmetric.

TRANSITIVITY. This is a little bit harder to express categorically, because is not a map from E or A, but from something else that even could not exist in the category. Transitivity is saying that the object T should have this pair of maps to A

$$
\begin{tikzcd}
& T \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
A & & A
\end{tikzcd}
$$

factoring through E as in the following diagram (the ordered pairs above each node are an example of how arrows operate)

(20)
$$
\begin{tikzcd}
& & & \overset{(a,b,d)}{T} \arrow[dl] \arrow[dr] & & & \\
& & \overset{(a,b)}{E} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & & \overset{(b,d)}{E} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & & \\
& \overset{(a)}{A} \arrow[drr] & & \overset{(b)}{A} \arrow[d] & & \overset{(d)}{A} \arrow[dll] & \\
& & & A & & &
\end{tikzcd}
$$

So the way to express transitivity is: the arrows $x_1, x_2 : E \longrightarrow A$ are transitive if for PB a pullback for the left diagram below, for all α, β

there is a k that makes the diagram on the right commute.

$$
\begin{tikzcd}
E \arrow[dr, "x_1"'] & & E \arrow[dl, "x_2"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & A \\
PB \arrow[u, dashed, "\exists k"] \arrow[ur, shift left, "\alpha"] \arrow[ur, shift right, "\beta"'] &
\end{tikzcd}
$$

What if your category does not posses pullbacks? In that case is harder to explain what transitivity means: You have to ask condition (20) for all test-maps

$$
\begin{tikzcd}
& T \arrow[d, dashed] \arrow[ddl, "\alpha"'] \arrow[ddr, "\beta"] & \\
& PB \arrow[dl] \arrow[dr] & \\
\dots & & \dots
\end{tikzcd}
\tag{21}
$$

in the upper square in the diagram (21). *Exercise:* Check that kernel pairs are transitive.

DISGRESSION.[2] A slighty different way of reformulating Transitivity is: Given a test object T an a pair of maps as in the diagram

$$
\begin{tikzcd}
T \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & E \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & A
\end{tikzcd}
$$

let us write (to easy notation):

$$
\begin{aligned}
x_1 \circ p &= a_1 \\
x_2 \circ p_1 &= a_2 \\
x_1 \circ p_2 &= a_3 \\
x_2 \circ p_2 &= a_4
\end{aligned}
$$

(the notation is motivated by the observation that if $E \subseteq A \times A$ and $T \subset E \times E$, the elements of T would be 4-tuples a_1, a_2, a_3, a_4).

Now, for all T, p_1, p_2

$$[a_2 = a_3] \Rightarrow \exists c : T \longrightarrow E$$

with $x_1 \circ c = a_1$ and $x_2 \circ c = a_4$.

EXERCISE.

[2]This is an aside that was given at the beginning of lecture 11 a propos of a comment about the material of this lecture. That is why we decided to include it here

1. This reformulated version of transitivity is equivalent to the previous one.
2. Each of the notions JM, R, S, T for $x_1, x_2 :\rightrightarrows$ is equivalent with its counterpart JM, R, S, T for

$$\mathcal{A}(-,E) \overset{x_1 \circ -}{\underset{x_2 \circ -}{\rightrightarrows}} \mathcal{A}(,E)$$

if and only if

$$\forall T \in |\mathcal{A}| : \mathcal{A}(T,E) \overset{x_1 \circ -}{\underset{x_2 \circ -}{\rightrightarrows}} \mathcal{A}(T,E)$$

is JM,R,S,T in the usual set-theoretic sense. (Hint: check first JM and then assuming it check R,S,T).

This is one more illustration of how the Yoneda representation is used to give idealized pictures: we could simply say

$$Y(E) \overset{Y(x_1)}{\underset{Y(x_2)}{\rightrightarrows}} Y(A)$$

is JM,R,S,T.

The Δ-algebra case. Let E, A be two Δ-algebras, and x_1, x_2 two Δ-algebra homomorphisms. Neccesary and sufficient condition for

$$E \overset{x_1}{\underset{x_2}{\rightrightarrows}} A \overset{f}{\dashrightarrow} ?$$

to be a kernel pair of some algebra homomorphism f is that

$$|E| \overset{x_1}{\underset{x_2}{\rightrightarrows}} |A|, \tag{22}$$

be the kernel pair of some function $|A| \longrightarrow ?$ (if and only if be JMRST).

Why? Just check carefully the conditions (for each property JMRST) on the corresponding diagrams above. For transitivity, the easiest way to see it is to notice that congruence relations are constructed set-theoretically.

Quotient Objects.(In general) Consider the diagram (22) and

$$f(a) = \{x \in |A| : x = x_2(e) \text{ where } e \in |E| \wedge x_1(e) = a\}$$

There is no reasonable way for $f(a)$ to be a Δ-algebra ($= \{x \in |A| : \exists p \in E : p = (a, x)\}$)

Let

$$Q = \{s \in P(|A|) : S = f(a) \wedge a \in |A|\}$$

We wish to make Q a Δ-algebra in such a way that f becomes a homomorphism:

$$\begin{tikzcd}
A^{n(\omega)} \arrow[r] \arrow[d, "\omega"'] & Q^{n(\omega)} \arrow[d, "\omega"] \\
|A| \arrow[r, "f"] & Q
\end{tikzcd}$$

Given $\omega \in \Delta$ and $n(\omega) \xrightarrow{(\cdots q_i \cdots)} Q$ we want: For all $a \in |A|^{n(\omega)}$, if $q_i = f(a_i)$, then $\omega(q) = f(\omega(a))$. This is our dream. How can we realize it?

If f is onto, how then to define $\omega(q)$? Fortunately there are tuples $a \in |A|^{n(\omega)}$ with $f(a_i) = q_i$ for each $i \in n(\omega)$, and (even more fortunately) if a and a' are two such

$$f(\omega(a)) = f(\omega(a'))$$

PROOF. If $f(a_i) = q_i$ and $f(a_i') = q_i$, that means $(a_i, a_i') \in |E|$, that is

$$p = (\cdots (a_i, a_i') \cdots) \in |E|^{n(\omega)}$$

$$x_1(\omega(p)) = \omega(x_1(p)) = \omega(\cdots a_i \cdots) = \omega(a)$$

and

$$x_2(\omega(p)) = \omega(x_2(p)) = \omega(\cdots a_i' \cdots) = \omega(a')$$

So $\omega(p) = (\omega(a), \omega(a'))$ and hence $f() = f()$. $\square$

This is a principle you see commonly in algebra (groups, rings, modules, boolean algebras, etc) where we have elements like $e, 0, \perp$ etc. by which you can recover all cosets form the corresponding (e-,e-, etc.) cosets by translation. This is not the general case as the following simple example in lattices shows:

Take the set of functions $f : \mathbb{R} \longrightarrow [0,1]$. Consider the operator σ (the support) $\sigma(f) = f^{-1}((0,1]) \subseteq P(\mathbb{R})$. Here for example, $[1]$ share its class with all functions, but $[0] = \{0\}$.

So it is very bad preparation for general algebra to start with modules, groups, rings, in which the kernel is a subobject. Kernels are subobjects in these categories, but this is not true in general.

LECTURE 10

Lawvere's Theorem (10/9)

The global picture. The last two lectures we have been a preparation for the formulation and proof of the characterization of equationally definable classes. I am going to give you the global picture of what is happening.

Given a Δ-algebra A $(= |A|, (\omega_A)_{\omega\in\Delta}))$, and functor (between $|A|$ and some sets)

$$X \underset{x_2}{\overset{x_1}{\rightrightarrows}} |A| \xrightarrow{\quad p \quad} Q$$

with p onto and x_1, x_2 JMRST, let us remark the following facts:

1. p is the coequalizer of (x_1, x_2) if and only if (x_1, x_2) is the kernel pair of p.
2. There is a unique way to make Q a Δ-algebra for which "p is a Δ-homomorphism"

$$\begin{tikzcd} |A|^{n(\omega)} \arrow[r, shift left, "p^{n(\omega)}"] \arrow[d, "\omega_A"'] & |Q|^{n(\omega)} \arrow[l, shift left, dashed, "\sigma^{n(\omega)}"] \arrow[d, "\omega_Q"] \\ |A| \arrow[r, shift left, "\text{onto}", "p"'] & Q \arrow[l, shift left, dashed, "\sigma"] \end{tikzcd}$$

in the long run

$$\omega = \omega \circ p^{n(\omega)} \circ \sigma^{n(\omega)} = p\omega \circ \sigma^{n(\omega)}$$

The same is true if you use ω'. That is, any particular operation has only one way to make Q a Δ-algebra.

3. There is a unique way to make Q a Δ-algebra for which "p is a Δ-homomorphism" if and only if

$$X \xrightarrow{(x_1, x_2)} |A| \times |A|$$

is a "subalgebra".

4. If both sides of (3) hold, then (1) is equally true at the level of Δ-algebras.
5. In particular, reading off parts of (1)-(4),

$$X \underset{x_2}{\overset{x_1}{\rightrightarrows}} A$$

in Δ-$\mathcal{A}lg$ is a kernel pair if and only if

$$|X| \underset{|x_2|}{\overset{|x_1|}{\rightrightarrows}} |A|$$

is a kernel pair of some function if and only if x_1, x_2 is JMRST.
6. An arbitrary Δ-algebra homomorphism $A \xrightarrow{p} Q$ (not necessarily surjective) is a coequalizer in Δ-$\mathcal{A}lg$ if and only if $|p|$ is onto.

REMARK 3. (5) and (6) (in this context perhaps) is what Emmy Noether saw as her first isomorphism theorem, in fact, (5) and (6) captures the heart of the FIT.

Backtracking over these six points: what if we have equations, i.e. we are in $(\Delta, \mathcal{E})$-$\mathcal{A}lg$ instead of Δ-$\mathcal{A}lg$? The answer is that everything works the same way, because every equation holds in the quotient class (some good reason missed here...) So we have the last remark:

7. For $(\Delta, \mathcal{E})$-$\mathcal{A}lg$ the situation is no different.

Now our goal is the following: In the diagram below, we have a functor $F_\Delta(-)$ with nice properties (recall lecture 7). We would like to do the same thing in the upper part of the diagram

$$\begin{tikzcd}
(\Delta, \mathcal{E})\text{-}\mathcal{A}lg \arrow[d, shift left, "U_\mathcal{E}"] \\
\Delta\text{-}\mathcal{A}lg \arrow[u, dashed, shift left, "F_\mathcal{E}(-)"] \arrow[d, shift left, "U_\Delta"] \\
\mathcal{S}ets \arrow[u, shift left, "F_\Delta(-)"]
\end{tikzcd}$$

'forget' the equations

'forget' the operations

i.e. a functor $F_\mathcal{E}(-)$ having the property

$$(\Delta, \mathcal{E})\text{-}\mathcal{A}lg(F_\mathcal{E}(A), -) \cong \Delta\text{-}\mathcal{A}lg(A, U_\mathcal{E}(-)) \cong U_\mathcal{E}^A(-)$$

(for the last $\cong$ recall $(U_\Delta)^k(-) = \mathcal{S}ets(k, U_\Delta(-)))$. How can we define $F_\mathcal{E}(A)$? For each $e = (e_1, e_2) \in \mathcal{E}$,

$$|A|^{n(\omega)} \underset{(e_2)_A}{\overset{(e_1)_A}{\rightrightarrows}} |A|$$

you want them to be equal,

$$|A|^{n(\omega)} \xrightarrow{e} |A| \times |A| \; (= |A \times A|)$$

and $\bigcup_{e \in \mathcal{E}} e(|A|^{n(\omega)}) \subseteq |A| \times |A|$ hence is a set. Let $\mathcal{F}$ be all RST subalgebras of $A \times A$ containing E. Let $E \subseteq X = \bigcap \mathcal{F}$. So define $F_\mathcal{E}(A)$ as this X.

How to see that this will work?

$$\begin{tikzcd} (\mathcal{E}) & A \arrow[d, "\Delta\text{-homo}"] \\ (e_1, e_2) \arrow[u] & T \quad (\text{a } (\Delta, \mathcal{E})\text{-alg.}) \end{tikzcd}$$

Whatever Δ-homomorphisms collapses together, it is guaranteed that it will collpases together e_1, e_2 too.

$$\begin{tikzcd} A \arrow[r, "\Delta\text{-hom}"] \arrow[d] & T \\ F_\mathcal{E}(A) = A/X \arrow[ur, "\exists!"'] & \end{tikzcd}$$

It is the same construction you do in abelian groups, etc.

All ingredients are available to say two of the three formulation or characterizations of equational definable classes:

(2) Lawvere's characterization of equationally definable classes of algebras as categories with explicit functors to $\mathcal{S}ets$.
(1) Birkhoff's HSP variety theorem.
(3) Historically the last, but the most general is Beck's theorem for varieties (over $\mathcal{S}ets$; over anything).

Lawvere's Theorem.

THEOREM 2 (Lawvere). *A category $\mathcal{A}$ along with a functor $U : \mathcal{A} \longrightarrow \mathcal{S}ets$ is "essentially" a variety (equationally definable class) of algebras if and only if*

1. *Each* $U^k(-)$ *is representable (for all* $k \in |\mathcal{S}ets|$*). (*$\mathcal{A}$ *has all "free" objects).*
2. $\mathcal{A}$ *has coequalizers and kernel pairs (of pretty much arbitrary single maps in* $\mathcal{A}$*).*
3. *(FIT)*
   (a) *A map* $p : A \longrightarrow Q$ *in* $\mathcal{A}$ is *a coequalizer (in* $\mathcal{A}$*) if and only if*

   $$U(p) \longrightarrow U(Q)$$

   is onto.

   (b) *A pair of maps in* $\mathcal{A}$

   $$X \overset{x_1}{\underset{x_2}{\rightrightarrows}} A$$

   is a kernel pair of maps in $\mathcal{A}$ *if*

   $$U(X) \overset{U(x_1)}{\underset{U(x_2)}{\rightrightarrows}} U(A)$$

   is a JMRST pair of functors.

There is a partial lie here: it is in "essentially" and "equationally definable class" (here refers to sets).

A MORE GENERAL CONCEPT OF VARIETY. Let us consider the category $(\Delta, \mathcal{E})$-$\mathcal{A}lg$, where Δ-ary class or operations ($n(\omega)$ a set, $\omega \in \Delta$), $\mathcal{E}$ still a class of equations.

The requirement to be a variety is that

$$\begin{tikzcd} (\Delta,\mathcal{E})\text{-}\mathcal{A}lg \arrow[d, shift right, "U"'] \\ \mathcal{S}ets \arrow[u, shift right, dashed, "\text{free functor } F"'] \end{tikzcd}$$

have a free functor with $F(k)$ representing $U^k(-)$.

Now, there are slightly more varieties. An example to convince (at least...) topologists:

$$\begin{tikzcd} \mathcal{K}\mathcal{T}_2 \arrow[d, shift right, "U"'] \\ \mathcal{S}ets \arrow[u, shift right, "F = \beta"'] \end{tikzcd}$$

where $\mathcal{KT}_2$ is the category of compact T_2 spaces and continuous maps, U is the underlying point set functor, and β the Stone-Čech compactification (for discrete spaces).

Recall that $U^k(A) = \mathcal{S}ets(k, U(A))$ and consider:

$$\begin{tikzcd} k_{disc} \arrow[r, "k \mapsto U(A)"] \arrow[dr] & A \\ & \beta_{disc} \arrow[u, "cont."'] \end{tikzcd}$$

Now take the smallest closest space....etc. As to (FIT), (a) and (b), ...COMPLETE THIS PART

So the theorem says that those topological spaces can be defined by operations and equations! What are these operations and equations? We will find out in the proof of Lawvere's theorem.

PROOF. (of Lawvere's theorem, idea) For the $\Leftarrow$ we have seen enough ...

In the other direction, (1),(2) and (3) implies that there is a variety on $\mathcal{A}$. Let Δ be all possible natural transformations $U^k \longrightarrow U$, for *all possible* k. It is better to organize them, so let us define the following category Θ_U:

$$|\Theta_U| = |\mathcal{S}ets|$$
$$\Theta_U(l, n) = \mathrm{nat}(U^n, U^l)$$

(here $|\mathcal{S}ets|$ plays the rol of the arity of the operations). There is a nice functor

$$\begin{array}{ccc} \mathcal{S}ets & \longrightarrow & \Theta_U \\ l \xrightarrow{f} n & \mapsto & U^f = - \circ f \end{array}$$

i.e. given an object A, define the map from $U^n(A) \longrightarrow U^l(A)$ as follows: take an element of $U^(A)$ (a map $n \xrightarrow{a} U(A)$) and compose it in the required form to get an element of U^l ($a \circ f$ is an l-tuple).

Now, Θ_U contains all we want ($U^n \longrightarrow U$ goes to $\Theta_U(1, n)$). □

Let us read for compact-T_2 spaces (i.e. in the category $\mathcal{HK}$) what we have so far: $\mathrm{nat}(U^k, U)$ are the k-ary operations on $\mathcal{HK}$. By Yoneda, it is isomorphic to $U(\beta(k))$. Oh!, ultrafilters on k. Take an ultrafilter $u \in U(\beta(k))$

$$\begin{tikzcd} k \arrow[d, "a"] \\ A \end{tikzcd} \qquad u(a) \in A$$

Some people would use the notation

$$(\hat{a})(u) = \lim_{\overrightarrow{i \in k}} a_i \in A$$

others, the presentation

$$\begin{tikzcd} u \in \beta(k) \arrow[d, dashed, "\hat{a}"'] & & k_{disc} \arrow[ll] \arrow[dll, "a"] \\ A & & \end{tikzcd}$$

So surprisingly Lawvere's theorem gives an equational presentation of compact T_2 spaces.

LECTURE 11

Beck's Theorem (10/14)

We need some definitions in order to state Beck's theorem.

SPLIT COEQUALIZERS. Take three objects and arrows as in

$$A' \;\underset{x_2}{\overset{x_1}{\rightrightarrows}}\; A \xrightarrow{p} A'' \qquad \delta: A \dashrightarrow A', \quad \sigma: A'' \dashrightarrow A \tag{23}$$

satisfying

$$p\,x_1 = p\,x_2 \tag{24}$$

$$p\,\sigma = \mathrm{id}_{A''} \tag{25}$$

$$x_1\delta = \mathrm{id}_A \tag{26}$$

$$x_2\delta = \sigma p \tag{27}$$

This four equations about the five maps in (23) defines a *split coequalizer situation.*

Some remarks: We wish p to be the equivalent in sets of being *onto*, that is why equation (24) is there. Also we want σ to be a kind of cross section for p, that is if A' and A'' were sets, σ is very much like choosing representatives of the partitions that defines p^{-1}. This is what equation (25) refers to. We also wish to say that δ is something that in $\mathcal{S}ets$ would mean $\delta(a) = (a, \sigma(p(a)))$, and this is the content of equations (26) and (27).

LEMMA 1. *Given a split coequalizer situation as in (23), p is in fact a coequalizer for (x_1, x_2).*

PROOF. Given map t as in the diagram satisfying $tx_1 = tx_2$,

$$\begin{tikzcd} A' \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & A \arrow[r, shift left, "p"] \arrow[dr, "t"'] & A'' \arrow[l, shift left, dashed, "\sigma"] \arrow[d, "z"] \\ & & T \end{tikzcd}$$

we must show that there is exactly one ($\exists^!$) z such that $z \circ p = t$. So we have to show:

a) $\exists^{\geq 1}$: try $z = t \circ \sigma$. Indeed,

$$\begin{aligned} z \circ p = (t \circ \sigma) \circ p &= t \circ (\sigma \circ p) \\ &= t \circ (x_2 \circ \delta) \\ &= (t \circ x_2) \circ \delta \\ &= (t \circ x_1) \circ \delta \\ &= t \circ (x_1 \circ \delta) \\ &= t \circ \mathrm{id}_A = t \end{aligned}$$

b) $\exists^{\leq 1}$: If $z \circ p = t$ and $\bar{z} \circ p = t$ we have

$$z \circ \mathrm{id}_{A''} = z \circ p \circ \sigma = t \circ \sigma = \bar{z} \circ p \circ \sigma = \bar{z} \circ \mathrm{id}_{A''}$$

$\square$

So, a split coequalizer situation gives a very 'algebraic' coequalizer. This does not happen in general (e.g. in groups). In the diagram

$$E \rightrightarrows \mathbb{Z} \longrightarrow \mathbb{Z}/(2)$$

where $E = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y - x \in 2\mathbb{Z}\}$, there is no way of getting a map (a group homomorphism) $\sigma : \mathbb{Z}/(2) \longrightarrow \mathbb{Z}$ such that equation (25) holds. To be a coequalizer in general is for more 'fluid' reasons that for 'algebraic' reasons.

Let us see what happens in $\mathcal{S}ets$. Consider the diagram

$$\begin{tikzcd} E \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & A \arrow[r, shift left, "p"] & A'' \arrow[l, shift left, dashed, "\sigma"] \\ & A \arrow[ul, "\delta"] \arrow[u, shift left, dashed, "\mathrm{id}_A"] \arrow[u, shift right, dashed, "\sigma p"'] & \end{tikzcd}$$

With the help of AC (axiom of choice) you can create a map $A \xleftarrow{\sigma} A''$. Hence we can get a pair of maps $\mathrm{id}_A, \sigma \circ p : A \longrightarrow A$ that come from one particular map p.

So, every time $x_1, x_2 : A \rightrightarrows A$ is a congruence relation in $\mathcal{S}ets$ and p is a coequalizer in $\mathcal{S}ets$, then the whole picture fits in a "split coequalizer situation".

LEMMA 2. *If*

$$A' \underset{x_2}{\overset{x_1}{\rightrightarrows}} A \xrightarrow{p} A''$$

is capable of fitting in a split coequalizer situation (in some category $\mathcal{A}$), then every *functor F from $\mathcal{A}$ to (a category) $\mathcal{B}$ winds up having*

$$F(A') \underset{F(x_2)}{\overset{F(x_1)}{\rightrightarrows}} F(A) \xrightarrow{F(p)} F(A'')$$

a coequalizer also (in fact split using $F(\sigma)$ and $F(\delta)$ where σ and δ are splitting data for the original $((x_1, x_2), p)$.

A module theorist would say: The roots of homology theory are the failure of functors hom and .... to preserve coequalizers. (In fact, homology theory is the measurement of how coequalizers are preserved by some functors).

EXAMPLE. Consider in the category of groups

$$\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : y - x \in 2\mathbb{Z}\} \underset{y}{\overset{x}{\rightrightarrows}} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/(2)$$

It is not a split coequalizer (as we indicated), but down, at the level of underlying sets we have σ, δ such that

$$\begin{tikzcd} |\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : y - x \in 2\mathbb{Z}\}| \arrow[r, shift left=2, "|x|"] \arrow[r, shift right=2, "|y|"'] & |\mathbb{Z}| \arrow[l, dashed, "\delta" description] \arrow[r, shift left, "|p|"] & |\mathbb{Z}/(2)| \arrow[l, dashed, shift left, "\sigma"] \end{tikzcd}$$

is a split coequalizer situation.

One of the miracles of algebra is ......

SPLIT COEQUALIZERS IN ALGEBRAS. Let $(\Delta, \mathcal{E})$-$\mathcal{A}lg$ be our focus. Suppose

$$A' \underset{x_2}{\overset{x_1}{\rightrightarrows}} A$$

is a pair of maps in $(\Delta, \mathcal{E})$-$\mathcal{A}lg$ for which there is Q, p, σ, δ making

$$\begin{tikzcd} U(A') \arrow[r, shift left=2, "U(x_1)"] \arrow[r, shift right=2, "U(x_2)"'] & U(A) \arrow[l, dashed, "\delta" description] \arrow[r, shift left, "p"] & Q \arrow[l, dashed, shift left, "\sigma"] \end{tikzcd} \tag{28}$$

a split coequalizer situation in $\mathcal{S}ets$.

PROPOSITION 6. *In the situation described above, there is a way and only one way to put an algebra structure on Q rendering p (see the diagram below) a Δ-homomorphism*

$$A' \underset{x_2}{\overset{x_1}{\rightrightarrows}} A \xrightarrow{''p''} (Q, (\omega)_{\omega \in \Delta})$$

in that Δ-$\mathcal{A}lg$ Q*, equations* $\mathcal{E}$ are *valid, and* p *is then a coequalizer of* (x_1, x_2) *in* $(\Delta, \mathcal{E})$-$\mathcal{A}lg$.

PROOF. For each $\omega \in \Delta$ consider the diagram defining the operations ω on Q:

(29)
$$\begin{tikzcd}[column sep=huge, row sep=huge]
(U(A'))^{n(\omega)} \arrow[r, shift left=2, "(Ux_1)^{n(\omega)}"] \arrow[r, shift right=2, "(Ux_2)^{n(\omega)}"'] \arrow[d, "\omega"'] & (U(A))^{n(\omega)} \arrow[l, dashed, "\delta^{n(\omega)}" description] \arrow[r, shift left, "p^{n(\omega)}"] \arrow[d, "\omega"'] & Q^{n(\omega)} \arrow[l, dashed, shift left, "\sigma^{n(\omega)}"] \arrow[d, "\omega"'] \\
U(A') \arrow[r, shift left=2, "Ux_1"] \arrow[r, shift right=2, "Ux_2"'] & U(A) \arrow[l, dashed, "\delta" description] \arrow[r, shift left, "p"] & Q \arrow[l, dashed, shift left, "\sigma"]
\end{tikzcd}$$

Define $\omega = p \circ \omega \circ \sigma^{n(\omega)}$. We see from the commutativity of the squares and from the hypothesis relatives to equation (28), that p is in fact a homomorphism, and in fact is the only way to put an algebra structure on Q with the required properties.

Also, p is homomorphism and onto, hence the equations $\mathcal{E}$ are valid (as we saw in lecture 8). The last thing is to check 'p is then a coequalizer of (x_1, x_2)'. Let us consider the diagram with $t\,x_1 = t\,x_2$,

$$\begin{tikzcd}
 & & T \\
A' \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & A \arrow[ur, "t"] \arrow[r, "p"'] & (Q, (\omega))
\end{tikzcd}$$

We have at the level of $\mathcal{S}ets$ $U(A) \xrightarrow{|t|} |T|$ in diagram (28). Hence we have in $\mathcal{S}ets$ a map f that makes the diagram below commute

$$\begin{tikzcd}
 & {|T|} \\
U(A) \arrow[ur, "|t|"] \arrow[r, "p"'] & Q \arrow[u, dashed, "f"']
\end{tikzcd}$$

Does f work at the right level? Just check

$$\begin{tikzcd}[row sep=huge]
Q^{n(\omega)} \arrow[r, "f^{n(\omega)}"] \arrow[d, "\omega"'] & {|T|^{n(\omega)}} \arrow[d, "\omega"] \\
Q \arrow[r, dashed, "f"] & {|T|}
\end{tikzcd}$$

$$\begin{aligned} f \circ \omega &= f \circ \underline{\omega \circ p^{n(\omega)}} \circ \sigma^{n(\omega)} \\ &= \underline{f \circ p} \circ \omega \circ \sigma^{n(\omega)} \\ &= \underline{|t| \circ \omega} \circ sigma^{n(\omega)} \\ &= \omega_T \circ \underline{t^{n(\omega)} \circ \sigma^{n(\omega)}} \\ &= \omega_T \circ f^{n(\omega)} \end{aligned}$$

□

Moral: algebra homomorphisms are just nicely behaved functions, not functions with something added!

One more definition in order to state Beck's theorem.

DEFINITION 7. Say

$$A' \underset{x_2}{\overset{x_1}{\rightrightarrows}} A$$

in $\mathcal{A}$ is *U-split* if there is p, σ, δ in $\mathcal{S}ets$ such that

$$\begin{tikzcd} U(A') \arrow[r, shift left=2, "U(x_1)"] \arrow[r, shift right=2, "U(x_2)"'] & U(A) \arrow[l, dashed, "\delta" description] \arrow[r, shift left, "p"] & U(A'') \arrow[l, dashed, shift left, "\sigma"] \end{tikzcd}$$

is a split coequalizer situation.

Finally we arrived to

THEOREM 3 (Beck). *Given category $\mathcal{A}$ and $U : \mathcal{A} \longrightarrow \mathcal{S}ets$, $\mathcal{A}$ (and U) is (are) a variety (and its underlying set functor) if and only if*

1. *Each U^k is representable.*
2. (a) *For all pair of maps*

$$A' \underset{x_2}{\overset{x_1}{\rightrightarrows}} A \qquad in\ \mathcal{A},$$

$\mathcal{A}$ has a coequalizer for x_1, x_2.

(b) *For all U-split*

$$A' \underset{x_2}{\overset{x_1}{\rightrightarrows}} A \qquad in\ \mathcal{A},$$

with coequalizer $A \xrightarrow{p} A''$ in $\mathcal{A}$,

$$U(A') \underset{U(x_2)}{\overset{U(x_1)}{\rightrightarrows}} U(A) \xrightarrow{U(p)} U(A'')$$

is a coequalizer diagram in $\mathcal{S}ets$.

(c) *For all U-split*

$$A' \overset{x_1}{\underset{x_2}{\rightrightarrows}} A \qquad \textit{in } \mathcal{A},$$

for all $p : A \longrightarrow Q$, *if* $U(p) = \mathrm{coeq}(Ux_1, Ux_2)$, *then* $p = \mathrm{coeq}(x_1, x_2)$.

Conditions (a), (b) and (c) seems very similar. Let us state the slogans behind them:

(a) $\mathcal{A}$ has coequalizers for U-split pairs.
(b) U preserves coequalizers of U-split pairs (i.e. if p is a coequalizer in $\mathcal{A}$, then $U(p)$ is a coequalizer in $\mathcal{S}ets$).
(c) U 'reflects' coequalizers of U-split pairs (i.e. if p is a coequalizer in $\mathcal{S}ets$, then it already was a coequalizer in $\mathcal{A}$).

There is no simple way of translating this conditions to the ones in Lawvere's theorem, so will give an independent proof.

REMARK 4. The marvelous thing about this theorem is that you could simple change once again the definition of variety, by replacing $\mathcal{S}ets$ by any other base category.

LECTURE 12

Adjoint Functors (10/16)

Today we are going to prove Lawvere's theorem. It is worth being prepared –just in case we need it– with some machinery.

Adjoint Functors. For each $k \in \mathcal{S}ets$, hte representabiliy of each functor

$$U^k : (\Delta, \mathcal{E})\text{-}\mathcal{A}lg \longrightarrow \mathcal{S}ets$$

amounts to the availability, for each $k \in |\mathcal{S}ets|$, of a $(\Delta, \mathcal{E})$-algebra $F_\Delta(k)$ such that

$$Y(F_\Delta(k)) \cong U^k$$

that is,

$$(\Delta, \mathcal{E})\text{-}\mathcal{A}lg(F_\Delta(k), -) \cong U^k(-)$$

or in other words, for all $A \in |(\Delta, \mathcal{E})\text{-}\mathcal{A}lg|$,

$$\Delta\text{-}\mathcal{A}lg(F_\Delta(k), A) \cong \mathcal{S}ets(k, U(A))$$

In *fact*, these isomorphisms *are* also natural in the $k \in \mathcal{S}ets$ variable.

LEMMA 3. *Suppose $\mathcal{A} \xrightarrow{Y} \mathcal{B}$ is a functor such that for each $A, A' \in \mathcal{A}$ we have the bijection*

$$\mathcal{A}(A', A) \xrightarrow[onto]{1-1} \mathcal{B}(YA', YA)$$

(in words: Y is "full and faithful" –also "fully faithful"). Then, given a functor

$$\mathcal{X} \xrightarrow{T} \mathcal{B}$$

for which it just so happens that, for each $X \in |\mathcal{X}|$ one can choose an $A_X \in |\mathcal{A}|$ and an isomorphism $Y(A_X) \cong T(X)$, the given choices thus made can (in one and only one way) be "factored up" into a functor

$$F : \mathcal{X} \longrightarrow \mathcal{A}$$

(with $F(X) = A_X$ for all $X \in |\mathcal{X}|$) whose composition $Y \circ F$ with Y is naturally isomorphic (via *the chosen isomorphisms) to T (see the diagram below).*

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "T"] \arrow[dr, dotted, "\exists F"'] & & \mathcal{B} \\ & \mathcal{A} \arrow[ur, "Y"'] & \end{tikzcd}$$

This is the natural counterpart of the situation in ....topological spaces, where from $S \cong X$ we lift to

$$(S, \tau_X|_S) \cong (X, \tau)$$

The above statement says no more than this.

PROOF. The construction of F: Given $\xi : X \longrightarrow Y$, we need to define $A_X \xrightarrow{F(\xi)} A_Y$. We know

$$\begin{tikzcd} Y(A_X) \arrow[r, "YF(\xi)"] \arrow[d, dash, "\cong"'] & Y(A_Y) \arrow[d, dash, "\cong"] \\ T(X) \arrow[r, "T(\xi)"'] & T(Y) \end{tikzcd}$$

So, going down-right-up, we have a map $YF(\xi) : Y(A_X) \longrightarrow Y(A_Y)$. Now, how to get one from $A_X \longrightarrow A_Y$? Simply using the bijection

$$\mathcal{A}(A_X, A_Y) \xrightarrow[\text{onto}]{1-1} \mathcal{B}(YA_X, YA_Y)$$

YF is naturally isomorphic to T: Given $X \xrightarrow{\xi} Y \xrightarrow{\zeta} Z$ we have the diagram

$$\begin{tikzcd} Y(A_X) \arrow[r, "YF(\xi)"] \arrow[d, dash, "\cong"'] & Y(A_Y) \arrow[r, "YF(\zeta)"] \arrow[d, dash, "\cong"] & Y(A_Z) \arrow[d, dash, "\cong"] \\ T(X) \arrow[r, "T(\xi)"'] & T(Y) \arrow[r, "T(\zeta)"'] & T(Z) \end{tikzcd}$$

we would like to understand is whether $Y(F(\zeta) \circ F(\xi)) =$

The other half of the battle: Why is F a functor? YF is a functor, so

$$YF(\zeta \circ \xi) = YF(\zeta) \circ YF(\xi) = Y(F(\zeta) \circ F(\xi))$$

hence ‘canceling’ Y (because faithfulness) we get

$$F(\zeta \circ \xi) = F(\zeta) \circ F(\xi)$$

Finally, $F(\mathrm{id}) = \mathrm{id}$ is much easier (use just faithfulness). $\square$

What good is that?

The passage from the functor $k \mapsto U^k$ in

$$\begin{tikzcd} \mathcal{S}ets \arrow[rr, "k \mapsto U^k"] \arrow[dr, "F"'] & & (\mathcal{S}ets^{\mathcal{A}})^{op} \\ & \mathcal{A} \arrow[ur, "Y \text{ (cov. Yoneda)}"'] & \end{tikzcd}$$

to the functor F, $k \mapsto F_{\Delta}(k)$ is just a particular case of a more general construction (cf. definition 8).

Let us reformulate again the conclusion of lemma 3 in a more general form:

Given $\mathcal{W} \subseteq |\mathcal{X}|$, assume the hypothesis of lemma 3. Then, given a functor

$$\mathcal{X} \xrightarrow{T} \mathcal{B}$$

for which it just so happens that, for each $X \in \mathcal{W} \subseteq |\mathcal{X}|$ one can choose an $A_X \in |\mathcal{A}|$ and an isomorphism $Y(A_X) \cong T(X)$, the given choices thus made can (in one and only one way) be “factored up” into a functor

$$F : \langle\mathcal{W}\rangle_{\mathcal{X}} \longrightarrow \mathcal{A}$$

($\langle\mathcal{W}\rangle_{\mathcal{X}}$ meaning the category with object-class $\mathcal{W}$ and $\hom(X, Y) = \mathcal{X}(X, Y)$, or in other words, the full subcategory of $\mathcal{X}$ with object class $\mathcal{W} \subseteq |\mathcal{X}|$) with $F(X) = A_X$ for all $X \in |\mathcal{X}|$, whose composition $Y \circ F$ with Y is naturally isomorphic (*via* the chosen isomorphisms) to $T|_{\langle\mathcal{W}\rangle_{\mathcal{X}}}$.

The proof is the same (just be careful to choose objects from $\mathcal{W}$).

It is this result that Albrecht Dold was looking at with his concept of *super-naturality.*

DEFINITION 8. Given functors

(30)
$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "U"] & \mathcal{X} \arrow[l, shift left, "F"] \end{tikzcd}$$

such that

(31)
$$\mathcal{A}(FX, A) \cong \mathcal{X}(X, UA)$$

is 1-1, onto and natural in both variables $X \in |\mathcal{X}|$ and $A \in |\mathcal{A}|$, then F is called the *right*[1] *adjoint* to U, and U is called the *left adjoint* to F.

Small remark: operator theory doesn't quite apply here: we could have a situation $F - U - G$ with F, U and U, G adjoints, but F and G with almost nothing in common.

EXAMPLES.

1. Let $\mathcal{A} = (\Delta, \mathcal{E})$-$\mathcal{A}lg$ and $\mathcal{X} = \mathcal{S}ets$. Then $F = F_{(\Delta,\mathcal{E})}$ and $U = U_{(\Delta,\mathcal{E})}$ is an adjoint pair of functors (an e.p. ...)
2. Define $F = \beta$ (the Stone-Čech compactification) and $U =$ "forget the topology". Then
$$\mathcal{KT}_2 \underset{\beta}{\overset{U}{\rightleftarrows}} \mathcal{S}ets$$
is a pair of adjoint functors.
3. Consider Think of $\mathcal{A}$ as $\langle$algebras satisfying $\mathcal{E}\rangle_{\Delta\text{-}\mathcal{A}lg}$, define $F =$"introduce as needed eqs. from $\mathcal{E}$". Then
$$(\Delta, \mathcal{E})\text{-}\mathcal{A}lg \underset{\beta}{\overset{U}{\rightleftarrows}} \Delta\text{-}\mathcal{A}lg$$
is a pair of adjoint functors.

Let us consider two important particular cases of equation (31),

(1) The case when $A = FX$ (for some $X \in \mathcal{X}$); we have
$$\begin{array}{ccc} \mathcal{A}(FX, FX) & \cong & \mathcal{X}(X, UFX) \\ \mathrm{id}_{FX} & \dashrightarrow & \eta_X \end{array}$$

Here we get, using the bijection, a map
$$X \xrightarrow{\eta_X} UFX \tag{32}$$
which is called the *front adjunction (unit)* to the adjointness (30).

(2) The case when $X = UA$:
$$\begin{array}{ccc} \mathcal{A}(FUA, A) & \cong & \mathcal{X}(UA, UA) \\ \varepsilon_A & \dashleftarrow & \mathrm{id}_{UA} \end{array}$$

Again, using the bijection, we get a map
$$FUA \xrightarrow{\epsilon_A} A \tag{33}$$
which is called the *back adjunction (counit)* to the adjointness (30).

[1] Mnemonic device: 'right' ('left') is meant to be which is to the right (left) of the comma.

EXERCISE. The maps in equations (32) and (33) are natural transformations. (There are two ways of proving it. (1) The intrincate: reprove Yoneda lemma in a particular case ... (2) Sophisticated proof: Yoneda is working behind this, take the identity, find the functor that is working here, etc.)

This is the approach that leads to monads. But let us came back to prove now –with this machinery– Lawvere's Theorem (see lecture 10).

PROOF. [Lawvere's Theorem][2] If $U : \mathcal{X} \longrightarrow \mathcal{S}ets$ has a left adjoint F, if $\mathcal{X}$ has a kernel pair (congruence relations of $\mathcal{X}$-isomorphisms) and coequalizers (of arbitrary pairs ... "difference kernels"), and

1. FIT_1 An $\mathcal{X}$-isomorphism p is a coequalizer if and only if U_p is onto.
2. FIT_2 A pair of $\mathcal{X}$-isomorphisms (x_1, x_2) constitutes a kernel pair if (by the way: and only if) (U_{x_1}, U_{x_2}) do.

Then $\mathcal{X}$ "looks just like" the $(\Delta, \mathcal{E})$-$\mathcal{A}lg$ category built up from $\Delta = \bigcup_{k \in |\mathcal{S}ets|} \mathrm{nat}(U^k, U) \times \{k\}$ and $\mathcal{E}$ (to be made precise in the case of the proof). Let Θ_U be the category defined as

$$|\Theta_U| = |\mathcal{S}ets|$$
$$\Theta_U(n,k) = \mathrm{nat}(U^k, U^n)$$

and as composition the usual composition of natural transformations. There is a functor φ

$$\begin{array}{ccc} \Theta_U & \xrightarrow{\quad\varphi\quad} & \mathcal{X} \\ n \xrightarrow{\alpha} k & \dashrightarrow & \varphi(n) \xrightarrow{\varphi(\alpha)} \varphi(k) \end{array}$$

where α "migrates" as follows:

$$\begin{array}{ccccccc} n & & Y\varphi(k) & & n & & \varphi(n) \\ \alpha\big\downarrow & \rightsquigarrow & \alpha\big\downarrow & \rightsquigarrow \alpha \in U^n(\varphi(k)) \rightsquigarrow & \alpha\big\downarrow & \rightsquigarrow_{adj.} & \alpha\big\downarrow \\ k & & U^n & & U\varphi(k) & & \varphi(k) \end{array}$$

Also there is a functor θ (that maps sets to sets)

$$\begin{array}{ccc} \mathcal{S}ets & \xrightarrow{\quad\theta\quad} & \Theta_U \\ n \xrightarrow{f} k & \dashrightarrow & - \circ f \text{ (or } U^f) \end{array}$$

[2]See section 4 in F.E.J. Linton, Some Aspects of Equational Categories.

so we have the following picture:

$$\begin{tikzcd} & \Theta_U \arrow[dr, "\varphi"] & \\ \mathcal{S}ets \arrow[ur, "\theta"] \arrow[rr, shift left, "F"] & & \mathcal{X} \arrow[ll, shift left, "U"] \end{tikzcd}$$

It is not difficult to check that F at the level of objects and arrows is the composition of the two previous functors we defined.

Now, Θ_U is in the middle ... so, let us redefine it directly:

$$|\text{Full image of } F| = |\mathcal{S}ets|$$
$$\{\text{Full image of } F\}(n,k) = \mathcal{X}(F(n), F(k))$$

It is easy to see that this is a category and recalling the proof of this we can check that it is precisely Θ_U. Hence we have two different ways of defining it!

So we arrived at the following situation:

$$\begin{tikzcd} \mathcal{X} \arrow[r, dashed, "\Phi"] \arrow[dr, "U"'] & \Theta_U\text{-}\mathcal{A}lg \arrow[r] \arrow[d] & \mathcal{S}ets^{(\Theta_U)^{op}} \arrow[d] \\ & \mathcal{S}ets \arrow[r, "Y"'] & \mathcal{S}ets^{\mathcal{S}ets^{op}} \end{tikzcd} \tag{34}$$

Given $X \in \mathcal{X}$, consider UX. How this can be a U-algebra ? Let $\lambda \in \text{nat}(U^k, U)$, and take

$$\lambda_X : (UX)^k \longrightarrow UX$$

Next thing to do: how this passage from X to λ_X manifest as a functor. The rest of the proof is now FIT_1 and FIT_2 makes Φ an equivalence of categories, and hence makes $\mathcal{X}$ an algebra. □

LECTURE 13

Proof of Lawvere's Theorem (10/21)

It came up several times unofficially: maybe is time to introduce it officially (it will be needed in the proof of Lawvere's Theorem).

PULLBACKS. Given (in any category) 3 objects and 2 maps

(35)
$$
\begin{tikzcd}
 & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}
$$

by a *pullback* for this diagram is meant an object P and two maps

$$
\begin{tikzcd}
P \arrow[r, "x"] \arrow[d, "y"'] & A \\
B &
\end{tikzcd}
$$

for which

1. $f \circ x = g \circ y$
2. For all pair of maps $T \xrightarrow{a} A$ and $T \xrightarrow{b} B$ satisfying $f \circ a = g \circ b$, there is a unique $T \xrightarrow{p} P$ solving the equations

$$x \circ p = a$$
$$y \circ p = b$$

All the information in one diagram looks as follows:

$$\begin{tikzcd}
T \arrow[dr, dotted, "p"] \arrow[drr, "a"] \arrow[ddr, "b"'] & & \\
& P \arrow[r, "x"] \arrow[d, "y"] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

P is an object we are seeking to characterize from the information of diagram (35) as

(36)
$$\mathcal{A}(T,P) \cong \{(a,b) \in \mathcal{A}(T,A) \times \mathcal{A}(T,B) : f \circ a = g \circ b\}$$

We seek $P \in |\mathcal{A}|$ with $Y(P) \in \mathcal{S}ets^{\mathcal{A}^{op}}$ isomorphic to (36).

$$\begin{tikzcd}
\text{Pullback} \arrow[r] \arrow[d] & \mathcal{A}(T,A) \arrow[d, "f \circ -"] \\
\mathcal{A}(T,B) \arrow[r, "g \circ -"'] & \mathcal{A}(T,C)
\end{tikzcd}$$

This is the set-theoretic version. In general, we ask "find me someone which job-description matches ... If we found one, as a corollary of the Yoneda Lemma it will satisfy the requirements.

Proof of Lawvere's Theorem (continued). Recall the diagram (34)at the end of last lecture.

Let us study in more detail the category Θ_U. The objects of Θ_U-$\mathcal{A}lg$ are pairs (A, a) such that

(37)
$$Y(A) = \mathrm{a} \circ \theta$$

This seems a little formalistic, but we will come back in a moment.

Maps from (A, a) to (B, b) are pairs $A \xrightarrow{\alpha} B$ such that $Y(f) = \alpha \circ \theta$. What does it mean to have a contravariant functor $\mathrm{a} : \Theta_U^{op} \longrightarrow \mathcal{S}ets$? For each n, k

(38)

$$
\begin{tikzcd}
n \arrow[d, "\omega"'] \arrow[dd, dashed, bend right, "\omega \circ \omega'"'] \arrow[r, mapsto] & \mathrm{a}(n) \\
k \arrow[d, "\omega'"'] \arrow[r, mapsto] & \mathrm{a}(k) \arrow[u, "\mathrm{a}(\omega)"'] \\
l \arrow[r, mapsto] & \mathrm{a}(l) \arrow[u, "\mathrm{a}(\omega')"'] \arrow[uu, dashed, bend right, "n(\omega \circ \omega') = \mathrm{a}(\omega) \circ \mathrm{a}(\omega')"']
\end{tikzcd}
$$

What is the statement (37)? The object-level interpretation of $Y(A) = \mathrm{a} \circ \theta$ is simply: $\mathrm{a}(n) = A^n$. So we have diagram (38) with $\mathrm{a}(i)$ replaced by A^i, for $i = n, k, l$. The mapping-level interpretation of $Y(f) = \alpha \circ \theta$ says: let $f : n \longrightarrow k$ (in $\mathcal{S}ets$); if we ask what $\mathrm{a}(\theta(f))$ is doing, $\mathrm{a}(\theta(f)) = - \circ f$. Why? Let us explain me $Y(f) = \alpha \circ \theta$ a little bit more first:

Mini Lemma. If

$$
\begin{aligned}
(f, \alpha) &: (A, \mathrm{a}) \longrightarrow (B, b) \\
(g, \alpha) &: (A, \mathrm{a}) \longrightarrow (B, b)
\end{aligned}
$$

then $f = g$ and $\alpha_n = - \circ f$.

Proof. If we establish $\alpha_n = - \circ f$, then $f = g$ follows. So let us prove this statement. Take $\alpha : \mathrm{a} \longrightarrow b$, hence $\alpha_n : \mathrm{a}(n) \longrightarrow b(n)$. But $\mathrm{a}(n) = Y(A)(n) = A^n$ (similarly for $b(n)$). Hence

$$\alpha_n = (\alpha \circ \theta)_n = (Y(f))_n = - \circ f$$

$\square$

So the question is that f *is more natural* than what you can expect. Now, back to the diagram

$$
\begin{tikzcd}
\mathrm{a}(n) & = & A^n \arrow[r, "f^n"] & B^n \\
\mathrm{a}(k) \arrow[u, "\mathrm{a}(f)"] & = & A^k \arrow[u, "\mathrm{a}(\omega)", "- \circ f"'] \arrow[r, "f^k"'] & B^k \arrow[u, "\mathrm{a}(\omega)"']
\end{tikzcd}
$$

The commutativity of this diagram is just the requirement that a is a natural transformation.

Mapping level interpretation of $(f, \mathrm{a}) : (A, \mathrm{a}) \longrightarrow (B, b)$.

I want to make it clear why Θ_U-$\mathcal{A}lg$ calls itself algebra. Θ-$\mathcal{A}lg$ is a $(\Delta, \mathcal{E})$-$\mathcal{A}lg$ where $\Delta = \bigcup_n \Theta(1, n) \times \{n\}$.

For $\omega \in \Theta(n,k)$ and $\mathrm{a}(\omega) : A^k \longrightarrow A^n$ we can write in a very algebraic notation

$$\mathrm{a}(\omega)(n) =_{def} a * \omega$$

and we have the equations

$$\begin{aligned} a * \mathrm{id}_k &= a && (39)\\ a * \theta(f) &= a \circ f && (40)\\ a * (\omega \circ \omega') &= (a * \omega') * \omega && (41) \end{aligned}$$

Equation (39) comes from $\mathrm{a}(\mathrm{id}_k) = \mathrm{id}_A(k)$. Equation (40) is $a\theta f = - \circ f$, and finally (41) comes from $\mathrm{a}(\omega' \circ \omega) = (\mathrm{a}(\omega) \circ \mathrm{a}(\omega))(a) = \mathrm{a}(\omega)(\mathrm{a}(\omega')(a))$. This are the set of equations $\mathcal{E}$.

Now, given $\mathcal{X}, U, F$, how to concret a $\Phi : \mathcal{X} \longrightarrow \Theta_U\text{-}\mathcal{A}lg$? If we consider the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[dr, dotted, "\Phi"] \arrow[drr, "Y(- \circ \varphi)"] \arrow[ddr, "U"'] & & \\ & \Theta_U\text{-}\mathcal{A}lg \arrow[r] \arrow[d] & \mathcal{S}ets^{\Theta^{op}} \arrow[d] \\ & \mathcal{S}ets \arrow[r, "Y"'] & \mathcal{S}ets^{\mathcal{S}ets^{op}} \end{tikzcd}$$

where the maps acts as follows

$$\begin{tikzcd} X \arrow[dr, dotted, "\Phi"] \arrow[drr] \arrow[dddr] & & \\ & ? \arrow[r] \arrow[dd] & \mathcal{X}(\varphi(-), X) \arrow[d] \\ & & \mathcal{X}(\varphi(\theta(-)), X) = \mathcal{X}(F(-), X) \\ & U(X) \arrow[r] & \mathcal{S}ets(-, UX) \end{tikzcd}$$

Now, the fact that F and U are adjoints tells us 'essentially' the equality $\mathcal{X}(F(-), X) = \mathcal{S}ets(-, UX)$ in the lower right corner. (Warning: F and U adjoints does not tell you this is an equality. We have to use additionally the definition of Θ_U in order to get it!). Hence, we have a pullback, $\Theta_U\text{-}\mathcal{A}lg$ and the *semantical comparison functor* Φ.

Under certain circumstances, Φ has a left adjoint $\check{\Phi}$

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift right, "\Phi"'] & \Theta_U\text{-}\mathcal{A}lg \arrow[l, shift right, "\check{\Phi}"'] \end{tikzcd}$$

Also a left adjoint F_Θ to $U_\Theta : \Theta_U\text{-}\mathcal{A}lg \longrightarrow \mathcal{S}ets$. The only thing we know going from $\mathcal{S}ets$ to $\mathcal{S}ets$ being fatten enough to be able to work as a left adjoint to U is .... Let us construct it. Consider

$$\begin{tikzcd} \mathcal{S}ets \arrow[dr, dotted, "F_\Theta"] \arrow[drr, "A \mapsto \Theta_U(-,\theta(A))"] \arrow[ddr, "F\circ U"'] & & \\ & \Theta_U\text{-}\mathcal{A}lg \arrow[r] \arrow[d, "U_\Theta"] & \mathcal{S}ets^{\Theta_U^{op}} \arrow[d] \\ & \mathcal{S}ets \arrow[r] & \mathcal{S}ets^{\mathcal{S}ets^{op}} \end{tikzcd} \tag{42}$$

Now, check that

$$\begin{aligned} \mathcal{S}ets(-,UFA) &= \mathcal{X}(F(-),FA) \\ &= \Theta_U(-,A) \\ &= \Theta_U(\theta(-),A) \end{aligned}$$

or simply

$$\mathcal{S}ets \xrightarrow{\theta} \Theta_U \xrightarrow{Y} \mathcal{S}ets^{\Theta_U^{op}}$$

Hence F_Θ is a pullback in diagram (42).
Exercise: F_θ is a left adjoint to U_Θ (hint: all the ingredients are on the table ...).

If and object of $\Theta_U\text{-}\mathcal{A}lg$ has the form $F_\Theta(A)$, then

$$\mathcal{X}(\check{\Phi}(F_\Theta(A)),X) \cong \Theta_U\text{-}\mathcal{A}lg(F_\Theta(A),\Phi(X)) \tag{43}$$
$$\cong \mathcal{S}ets(A, U_\Theta\Phi(X)) \tag{44}$$
$$\cong \mathcal{S}ets(A, UX) \tag{45}$$
$$\tag{46}$$

Remarks: Isomorphism (43) is natural in $\mathcal{X}$, the line (44) is because F_Θ, U_Θ are adjoints, and (45) is due to the fact $U_\Theta\Phi = U$.

We can say: to the extend we can at least find values for $F(A)$, we have at least a partial left adjoint. Now, how can we extend this from the algebra $F(A)$ to all algebras in Θ_U-$\mathcal{A}lg$? Observing that Θ_U-$\mathcal{A}lg$ is a quotient by a decent relation.

$A \in |\Theta_U\text{-}\mathcal{A}lg|$, $F_\Theta(U_\Theta(A) \mapsto A$ which at the level of objects is surjective.

$$F_\Theta(UE) \longrightarrow E \rightrightarrows F_\Theta(U_\Theta(A)) \longrightarrow A$$

Φ has left adjoint $\check{\Phi}$ if $\mathcal{X}$ has coequalizers[1].

[1]Historical comment: Looking at the adjoints

$$\mathcal{S}ets \underset{U_\Theta}{\overset{F_\Theta}{\rightleftarrows}} \Theta_U\text{-}\mathcal{A}lg$$

and

$$F \circ U : \mathcal{S}ets \longrightarrow \mathcal{S}ets$$

one is puzzled by the question: Why did Lawvere didn't discover monads? Perhaps he knew Eilenberg- .. was working with them, that ..... Perhaps he prefered to concentrate in the category Θ_U-$\mathcal{A}lg$ leaving the surrounding work to other people ...

LECTURE 14

Continuation of the proof (10/23)

We have

$$\begin{tikzcd}
\mathcal{X} \arrow[r, shift left, "\Phi"] \arrow[d, shift left, "U"] & \Theta_U\text{-}\mathcal{A}lg \arrow[l, shift left, "\check{\Phi}"] \arrow[d, shift left, "U_\theta"] \\
\mathcal{S}ets \arrow[u, shift left, "F"] \arrow[r, equal] & \mathcal{S}ets \arrow[u, shift left, "F_\theta"]
\end{tikzcd}$$

where $|\Theta_U| = \mathcal{S}ets$ and for the maps we have many choices

$$\Theta_U(n,k) = \begin{cases} 1.\ \mathrm{nat}(U^k, U^n) \\ 2.\ \mathcal{X}(F(n), F(k)) \\ 3.\ \mathcal{S}ets(n, UF(k)) \\ 4.\ \mathcal{S}ets(n, \mathrm{nat}(U^k, U)) \\ 5.\ U^n(F(k)) \end{cases}$$

all of which are the same: 1 and 4 are the same: $(U^n)_X = (U_X)^n$. The adjointness between F and U gives the equivalence between 2 and 3. For 1 and 5, observe that U^k is representable and represented by $F(k)$ (Yoneda Lemma). The equality of 3 and 6 is by definition of what is U^n. Finally 3 and 4 follows from Yoneda Lemma (represented by $F(k)$). This is the description of the category Θ_U-$\mathcal{A}lg$.

Now let us see the functor $\Phi : \mathcal{X} \longrightarrow \Theta_U\text{-}\mathcal{A}lg$,

$$\Phi(X) = (UX, \{(UX)^n \xrightarrow{\omega_X} (UX)^n\}_{\omega \in \Delta})$$

where $U^k \xrightarrow{\omega} U^n$ and Δ is defined as all posible ....

The

algebraic language		nat. transf. language
$a * \omega$	$=$	$\omega_X(a)$

For $X \xrightarrow{\xi} Y$, $U(\xi) : \Phi(X) \longrightarrow \Phi(Y)$ is a homomorphism because

$$\xi \circ (a * \omega) = (\xi \circ a) * \omega$$

(This is the reason why many algebraists, f ex. Jacobson, prefer to write function in one side and operations on the other).

SOME MOTIVATION FOR THE LEFT ADJOINT $\check{\Phi}$.

PROPOSITION 7. 1. *Every functor that* is *a left adjoint preserves coequalizers (much more too ...)*
2. *Every functor that* is *a right adjoint preserves kernel-pairs (products, pullbacks, in general limits –we will see it later–, but this is what we need now).*

Observe

$$\begin{aligned} \mathcal{X}(F_n, X) &\cong \mathcal{S}ets(n, UX) \\ &= \mathcal{S}ets(n, U_\theta(\Phi(X))) \end{aligned}$$

Using left adjoint F_θ

$$\cong \Theta_U\text{-}\mathcal{A}lg(F_\theta(n), \Phi(X))$$

For what? Morphisms $\mathcal{X}(?, X)$ looks like last line above. If there were a left adjoint to $\Phi(X)$ we would have

$$\cong \mathcal{X}(\check{\Phi}(F_\theta(n)), X)$$

So, for values $\check{\Phi}(F_\theta(n))$ we use $F(n)$.

PROOF. (1) Take a pair of maps

$$A' \underset{y}{\overset{x}{\rightrightarrows}} A \xrightarrow{p = \text{coeq}(x,y)} A''$$

Applying the functor F, we get

$$FA' \underset{Fy}{\overset{Fx}{\rightrightarrows}} FA \xrightarrow{Fp} FA''$$

We want to check Fp is a coequalizer. Given $t : FA \longrightarrow T$ with $Fx = t \circ Fy$, find matching $\hat{t} : A \longrightarrow UT$. It is easy to check $\hat{t}x = \hat{t}y$. So, by hypothesis we have $\hat{t}$ factors! through A'', that is, $\exists! w$ with $qp = \hat{t}$. But then $\check{q} \circ Fp = t$.

(2) works essentially in the same way.

$$\begin{tikzcd} A' \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & A \arrow[r, "f"] & A'' \\ FE \arrow[u, dotted, "p"] \arrow[ur, dotted, shift left, "\hat{a}"] \arrow[ur, dotted, shift right, "\hat{b}"'] & & \end{tikzcd}$$

$$\begin{tikzcd} UA' \arrow[r, shift left, "Ux"] \arrow[r, shift right, "Uy"'] & UA \arrow[r, "Uf"] & UA'' \\ E \arrow[u, dotted, "\check{p}"] \arrow[ur, dotted, shift left, "a"] \arrow[ur, dotted, shift right, "b"'] & & \end{tikzcd}$$

Given a, b with $Uf \circ a = Uf \circ b$, the compositional requirement

$$xp = \hat{a} \qquad yp = \hat{b}$$

translate in

$$Ux\check{p} = a \qquad Uy\check{p} = b \tag{47}$$

So there is $\check{p}$ with (47). $\square$

Known how $\check{\Phi}$ needs to treat the $F_\theta(n)$'s and knowing that it is a left adjoint, $\check{\Phi}$ will have to preserve coequalizers. We will know how to define

$$\check{\Phi}(\mathrm{coeq}(F_\theta(n) \overset{x}{\underset{y}{\rightrightarrows}} F_\theta(k))) \quad = \mathrm{coeq}(F(n) \overset{\check{\Phi}(x)}{\underset{\check{\Phi}(y)}{\rightrightarrows}} F(k))$$

So, why can every Θ_U-algebra be expressed in this way? Take $A \in \Theta_U$-$\mathcal{A}lg$. For each natural transformation $U^k \xrightarrow{\omega} U^n$ we have

$$|A|^k \xrightarrow{-*\omega} |A|^n$$

Now,

$$F_\theta(|A|) \xrightarrow{\varepsilon_A} A$$
$$|A| \xrightarrow{\mathrm{id}} |A| = U_\theta(A)$$

This function at the level of sets is "very surjective"

$$
\begin{tikzcd}
{|A|} \arrow[r, "\eta_{|A|}"] \arrow[rr, dashed, bend right, "\mathrm{id}_{|A|},\text{ or better } U(\mathrm{id}_A)"'] & {|F_\theta(|A|)|} \arrow[r, "U_\theta(\varepsilon_A)"] & {|A|}
\end{tikzcd}
$$

This can be done no matter what algebra you are looking at. So take

$$
\begin{tikzcd}
F_\theta(|E_A|) \arrow[r, "\varepsilon_{E_A}"] \arrow[rr, dashed, bend left, "a"] \arrow[rr, dashed, bend right, "b"'] & E_A \arrow[r, shift left] \arrow[r, shift right, "\textit{kernel pair}"'] & F_\theta(|A|) \arrow[r, "\varepsilon_A"] & A
\end{tikzcd}
\tag{48}
$$

What is the coequalizer of a, b? It is ε_A. The diagramatic-style proof is: Given test maps t with $ta = tb$, i.e. $tx\varepsilon_{E_A} = ty\varepsilon_{E_A}$

$$
\begin{tikzcd}
 & & T \\
F_\theta(|E_A|) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & F_\theta(|A|) \arrow[ur, "t"] \arrow[r, "\varepsilon_A"] & A \arrow[u, dashed]
\end{tikzcd}
$$

Use heavily ....

DEFINITION 9. (if we can (!) .... coequalizerls had better *exists* in $\mathcal{X}$)

$$\check{\Phi}(A) = \mathrm{coeq}_{\mathcal{X}}(\check{\Phi}(x \circ \varepsilon_{E_A}), \check{\Phi}(y \circ \varepsilon_{E_A}))$$

Recall () that every time you have an adjoint, you have a back adjunction

$$\check{\Phi}(\Phi X) \xrightarrow{\beta} X \tag{49}$$

and also, a front adjunction

$$A \xrightarrow{\alpha} \Phi(\check{\Phi} A) \tag{50}$$

The next trick is to see that the three ingredients in FIT make each (49) and (50) an isomorphism.

LECTURE 15

(10/28)

Leftover from last time: if an adjoint pair is given, and

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, shift left, "\Phi"] \arrow[dr, shift left, "U"] & & \Theta_U\text{-}\mathcal{A}lg \arrow[ll, shift left, "\check{\Phi}"] \arrow[dl, shift left, "U_\theta"] \\
& \mathcal{S}ets \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F_\theta"] &
\end{tikzcd}$$

is formed "as usual", with left adjoint F_θ for U_θ, then

$$\Phi \circ F \cong F_\theta$$

(Last clase we couldn't see the proof, that was at the other side of the window ... we just needed to unravel the shade ...)

Recall the functor U_θ was defined by

$$\Theta_U\text{-}\mathcal{A}lg \longrightarrow \mathcal{S}ets^{\Theta_U^{op}} \xrightarrow{restr.} \mathcal{S}ets^{\mathcal{S}ets^{op}} \xrightarrow{Y} \mathcal{S}ets$$

We have to remember how we constructed F_θ; First

$$\Phi(F(n))\{k\} = U^k(F(n)) \tag{51}$$

(this is essentially how the passage from $\mathcal{X}$ to $\mathcal{S}ets^{\Theta_U^{op}}$ works). On the other hand,

$$F_\theta(n)(k) = \Theta_U(k, n) \tag{52}$$

Now, $\Theta_U(k, n)$ had at least 5 different expressions, one is $\mathrm{nat}(U^n, U^k)$, and by Yoneda it is isomorphic to $U^K(F(N))$. So, by equations (51) and (52), at the level of objects, the functors $\Phi(F(n))\{k\}$ and $F_\theta(n)(k)$ are the same.

Exercise: Check the map-level.

This help explaining why $\check{\Phi}$ at the level of free algebras is a perfect inverse of Φ all the time. So we could perhaps form

$$\mathcal{G} = \{\Theta_U\text{-}\mathcal{A}lg : \mathcal{X}(''\check{\Phi}(A)'', -) \cong \Theta_U\text{-}\mathcal{A}lg(A, \Phi(-))\}$$

i.e., $\mathcal{G}$ are those Θ_U-algebras for which the functor

$$\Theta_U\text{-}\mathcal{A}lg(A, \Phi(-)) : \mathcal{X} \longrightarrow \mathcal{S}ets$$

is representable by an object "$\check{\Phi}$" in $\mathcal{X}$. We can say that at least for the elements of $\mathcal{G}$ we have $\check{\Phi}$ is adjoint to Φ, and the supernaturality lemma tells us that the extension

$$\check{\Phi} : \langle \mathcal{G} \rangle_{\Theta_U} \longrightarrow \mathcal{X}$$

What we know at this point is that $\mathcal{G}$ contains all the free Θ_U-algebras. Now, since *every* Θ_U-$\mathcal{A}lg$ is a coequalizer of some pair of maps between free algebras, Θ_U-$\mathcal{A}lg \subseteq \mathcal{G}$ if $\mathcal{X}$ has coequalizers (actually is enough to have "enough coequalizers") and the reason is that $\check{\Phi}$ must preserve coequalizers.

Standing assumption for today from now on:

1. U has a left adjoint F
2. $\mathcal{X}$ has coequalizers (this assumes $\check{\Phi}$ is "everywhere defined")
3. $U(x)$ is epi $\Rightarrow$ x is coequalizer
4. $U(x)$ is epi $\Leftarrow$ x is coequalizer
5. $\mathcal{X}$ has kernel pairs

Recall a general functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ is faithful if each function

$$T_{A,A'} : \mathcal{A}(A, A') \longrightarrow \mathcal{B}(TA, TA') \quad \text{is 1-1}$$

Why we rise this? Because U is going to be faithful iff Φ is faithful, and the proof is somehow given by the fact that Yoneda is faithful.

CLAIM. U_θ is always faithful (Why?: because of definition of algebras). And so, Φ is faithful $\Rightarrow$ $U_\theta \circ \Phi$ is faithful $\Rightarrow$ U is faithful.

Conversely, U is faithful, then Φ is faithful.

PROOF. (of claim) Remembering that $U(x)$ *is* a coequalizer (in $\mathcal{S}ets$, i.e. onto) iff x itself is a coequalizer in $\mathcal{X}$, let us show that U is faithful. Take

$$X \overset{a}{\underset{b}{\rightrightarrows}} Y$$

and assume that $Ua = Ub$ in

$$UX \overset{Ua}{\underset{Ub}{\rightrightarrows}} UY$$

and prove $a = b$. Consider

$$FUX \xrightarrow{\varepsilon_X} X \overset{a}{\underset{b}{\rightrightarrows}} Y$$

So we have at the level of $\mathcal{S}ets$

$$\begin{tikzcd}
UFUX \arrow[dr, "U\varepsilon_X"] & & \\
UX \arrow[u, "\eta_{UX}"] \arrow[r, "\mathrm{id}"'] & UX \arrow[r, shift left, "Ua = Ub"] \arrow[r, shift right] & UY
\end{tikzcd}$$

hence $a \circ \varepsilon_X = b \circ \varepsilon_X$. *But,* $U(\varepsilon_X)$ is "very" surjective! So ε_X is a coequalizer. Hence ε_X is an "epimorphism", meaning that $a \circ \varepsilon_X = b \circ \varepsilon_X \Rightarrow a = b$ is valid. $\square$

In fact, calling a map e *epimorphism* if

$$\forall a, b : A \rightrightarrows B \quad (a \circ e = b \circ e \Rightarrow a = b)$$

we proved a more general fact:

LEMMA 4. *A functor U that has a left adjoint and that satisfies $U(x)$ epi $\Rightarrow x$ is epi,* is *faithful.*

So, with (2) and (3) Φ is faithful. With the help of (1), (2) and (3) together, Φ is provably full (as is $\check{\Phi}$!).

Anyway, Φ acts "full" so far as *maps* between free things

(53)
$$\begin{tikzcd}
\mathcal{X}(F(n), F(k)) \arrow[r, "\Phi"] \arrow[d, equal] & \Theta_U\text{-}\mathcal{A}lg(\Phi F(n), \Phi F(k)) \arrow[d, equal] \\
\mathcal{X}(\check{\Phi} F_\theta(n), F(k)) \arrow[r, "\cong"] & \Theta_U\text{-}\mathcal{A}lg(F_\theta(n), F_\theta(k)) \arrow[d, equal] \\
\mathcal{X}(\check{\Phi} F_\theta(n), \check{\Phi} F_\theta(k)) & \Theta_U\text{-}\mathcal{A}lg(F_\theta(n), F_\theta(k)) \arrow[l, "\check{\Phi}"]
\end{tikzcd}$$

Everything works except we don't know if Φ is well defined. How can this happen if not because Φ and $\check{\Phi}$ are bijections and inverse one of the other?

Let us prove Φ is full: given $X, Y \in \mathcal{X}$ and $\Phi X \xrightarrow{f} \Phi Y$ seek $Y \xrightarrow{\alpha} Y$ in $\mathcal{X}$ with $\Phi(\alpha) = f$.

$$
\begin{tikzcd}
\equiv_{\varepsilon_X} \arrow[d, shift right, "x_1"'] \arrow[d, shift left, "x_2"] & \equiv_{\varepsilon_Y} \arrow[d, shift right, "x_1"'] \arrow[d, shift left, "x_2"] \\
FUX \arrow[r, "F(U_\theta f)"] \arrow[d, "\varepsilon_X"'] \arrow[dr] & FUY \arrow[d, "\varepsilon_Y"] \\
X \arrow[r, dotted, "\alpha"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
U_\theta \Phi X \arrow[r, "U_\theta f"] \arrow[d, equal] & U_\theta \Phi Y \arrow[d, equal] \\
UX \arrow[r, "U_\theta f"'] & UY
\end{tikzcd}
$$

("sequential switch" diagram: taking x_1 (x_2) with the left (right) arrow the diagram commutes). The goal is to figure out why

$$F(U_\theta f) \circ x_1 = F(U_\theta f) \circ x_2$$

The handy thing to use now is hypothesis (5) [1]. The diagonal dotted arrow: is enough to see that

$$U(\alpha) = U_\theta \Phi(\alpha) = U_\theta(f) \tag{54}$$

Exercise Prove equation (54) above.

The last thing is to see is comparing

$$
\begin{tikzcd}
\check{\Phi}\Phi X \arrow[d, "\beta_X"'] & \Phi\check{\Phi}(A) \arrow[d, "\alpha_A"] \\
X & A
\end{tikzcd}
$$

What does it take to see that this is an isomorphism?: next class.

[1]All this is ill motivated if you don't know about monads or triples.

LECTURE 16

In where the final part of the proof is given, and other interesting stories are introduced (10/30)

Our setting now: still

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, shift left, "\Phi"] \arrow[dr, shift left, "U"] & & \Theta_U\text{-}\mathcal{A}lg \arrow[ll, shift left, "\check{\Phi}"] \arrow[dl, shift left, "U_\theta"] \\
& \mathcal{S}ets \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F_\theta"] &
\end{tikzcd}$$

where

Θ_U "=" (natural transformations among $\{\mathcal{S}ets\}$-indexed powers of $U)^{op}$.

"=" ($\mathcal{X}$-morphisms among $\{\mathcal{S}ets\}$-indexed values of F).

and $\mathcal{X}$ with coequalizers, kernel pairs, $U(p)$ onto iff p is a coequalizer, and ...

Additional new hypothesis for today will be ...some reference here! We already saw F is full, faithfull, has a left adjoint, and $U_\theta \circ \Phi = U$, $\Phi \circ F = F_\theta$.

The only thing we need to see today is that $\Phi\check{\Phi}$ is "close" to the identity in Θ_U-$\mathcal{A}lg$.

So, let us assume (*). Given $A \in |\Theta_U|$, $\check{\Phi}(A)$ is a coequalizer in $\mathcal{X}$, namely the one we got by noticing that in Θ_U-$\mathcal{A}lg$:

(55)
$$\begin{tikzcd}
F_\theta U_\theta E \arrow[r, "\varepsilon_E"] \arrow[rr, dotted, bend left, "x_1\varepsilon_E"] \arrow[rr, dotted, bend right, "x_2\varepsilon_E"'] & E \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & F_\theta(U_\theta A) \arrow[r, "\varepsilon_A"] & A
\end{tikzcd}$$

and at the level of $\mathcal{S}ets$:

$$\begin{tikzcd}
U_\theta E \arrow[r, shift left, "U_\theta x_1"] \arrow[r, shift right, "U_\theta x_2"'] & U_\theta F_\theta U_\theta(A) \arrow[r, "U_\theta \varepsilon_A"] & U_\theta(A)
\end{tikzcd}$$

The counterpart to this (applying F) is

$$FU_\theta E \overset{\bar{x_1}}{\underset{\bar{x_2}}{\rightrightarrows}} FU_\theta(A) \xrightarrow{\text{coeq}} \check{\Phi}(A) \tag{56}$$

i.e. $x_i\varepsilon_E$ are values of Φ and with them get the $\bar{x}_i$.

Now apply Φ to (56):

$$\Phi(FU_\theta E) \overset{\Phi(\bar{x_1})}{\underset{\Phi(\bar{x_2})}{\rightrightarrows}} \Phi(FU_\theta(A)) \xrightarrow{\Phi(\text{coeq})} \Phi\check{\Phi}(A) \tag{57}$$

Considering diagrams (55) and (57) together we arrive to the following situation:

$$\begin{tikzcd}
F_\theta U_\theta E \arrow[r, shift left, "x_1\varepsilon_E"] \arrow[r, shift right, "x_2\varepsilon_E"'] \arrow[d, dotted, no head, "="] & F_\theta(U_\theta A) \arrow[r, "\varepsilon_A"] \arrow[d, dotted, no head, "="] & A \arrow[d, dotted, "\alpha_A"] \\
\Phi(FU_\theta E) \arrow[r, shift left, "\Phi(\bar{x_1})"] \arrow[r, shift right, "\Phi(\bar{x_2})"'] & \Phi(FU_\theta(A)) \arrow[r, "\Phi(\text{coeq})"] & \Phi\check{\Phi}(A)
\end{tikzcd} \tag{58}$$

Using the two equalities we got a map $\alpha_A : A \longrightarrow \Phi\check{\Phi}A$. Now, because Φ is an adjoint, preserves kernel pairs. Hence $\Phi(\text{coeq})$ is a coequalizer ... Now the only thing left is:

Exercise: Prove that α_A is isomorphism (this problem could be hard).

So, finally we arrived to the end of the story:

$$\begin{tikzcd}
\mathcal{X} \arrow[r, shift right, "\Phi"'] \arrow[r, phantom, "\text{fully faithful}"] & \mathcal{A}lg \arrow[l, shift right, "\check{\Phi}"']
\end{tikzcd}$$

for every $X \in \Theta_U\text{-}\mathcal{A}lg$,

$$X \cong \check{\Phi}\Phi(X) \qquad\qquad \Phi(X) \cong \Phi(\check{\Phi}(\Phi(X)))$$

Introduction to Monads. One of the things we could have focus in is the iterated repetition of UF (in some sense that is the focus of Beck's theorem).

$$\begin{tikzcd}
\mathcal{X} \arrow[dr, shift left, "U"] & \\
& \Omega \arrow[ul, shift left, "F"]
\end{tikzcd}$$

Recall the particular cases of the equation

$$\mathcal{A}(A, UX) \cong \mathcal{X}(FA, X)$$

we studied in lecture 12, after definition 8. For $X = FA$ we get:

$$A \xrightarrow{\eta_A} UFA \leftrightarrow \mathrm{id}_{FA}$$

and for $A = UX$ we get

$$\mathrm{id}_{UX} \quad \leftrightarrow \quad FUX \xrightarrow{\varepsilon_X} X$$

We could define two maps h, r as in the following diagram

$$\begin{tikzcd}
& \mathcal{X}(FA, FUX) \arrow[dr, "\varepsilon_X \circ -"] & \\
\mathcal{A}(A, UX) \arrow[ur, "F_{A,UX}"] \arrow[rr, dotted, shift left, "h_{A,X}"] & & \mathcal{X}(FA, X) \arrow[ll, dotted, shift left, "r_{A,X}"] \arrow[dl, "U_{FA,X}"] \\
& \mathcal{A}(UFA, UX) \arrow[ul, "- \circ \eta_A"] &
\end{tikzcd}$$

We have the following important observations:

LEMMA 5. 1. *For all* $f \in \mathcal{A}(A, UX)$

$$\varepsilon_X \circ F(f) = h_{A,X}(f)$$

2. *In the other direction, for all* $\varphi \in \mathcal{X}(FA, X)$,

$$U(\varphi) \circ \eta_A = r_{A,X}(\varphi)$$

PROOF. Just a naturality check, virtually a fragment of the proof of the Yoneda Lemma. Simple Exercise. □

In the Yoneda Lemma, for any contravariant functor $T : \mathcal{A}^{op} \longrightarrow \mathcal{S}ets$, and for any object $A \in \mathcal{A}$,

$$TA \cong \mathrm{nat}(\mathcal{A}(-, A), T)$$

The question could arise, given $\lambda \in \mathrm{nat}(\mathcal{A}(-, A), T)$ and consider its image $\lambda_A(\mathrm{id}_A) \in TA$ under the bijection. When does an element of TA guarantee that T is representable by $\mathcal{A}(-, A)$?

DEFINITION 10. In the situation above, $a \in TA$ is called *universal element* if, under Yoneda Lemma, $a = \lambda_A(\mathrm{id}_A)$ for some $\lambda : \mathcal{A}(-, A) \xrightarrow{\cong} T$.

What does it mean more exactly?: A universal element is a kind of "generic variable".

PROPOSITION 8. *$a \in TA$ is universal iff $\forall X \in |\mathcal{A}|, \forall x \in TX, \exists! \xi$ solution to the equation*

$$T(\xi)(a) = x \qquad (\xi \in \mathcal{A}(X, A)) \tag{59}$$

PROOF. Suppose a is universal, use the isomorphism $\lambda_X : \mathcal{A}(X, A) \xrightarrow{\cong} TX$. So given $x \in TX$, we have $T(\lambda_X^{-1}(x))(a) = x$, hence consider $\xi = \lambda_X^{-1}(x)$.

In the other direction, if there is always a unique solution of equations of the sort (59), want to show that $\lambda : \mathcal{A}(-, A) \longrightarrow T$ is an equivalence. What λ does: $\lambda(\xi) = T(\xi)(a)$. The uniqueness of the solution to (59) is telling you that $\lambda_X : \mathcal{A}(X, A) \longrightarrow TX$ is 1-1 and onto. $\square$

This is at the bottom of the statements of Lemma 5. It is just expresed in a more intrincated way. As a result of this extra complication, we get something. We can think of all η together as a map

$$\mathrm{id}_{\mathcal{A}} \xrightarrow{\ \eta\ } UF$$

The same with ε:

$$\mathrm{id}_{\mathcal{X}} \xleftarrow{\ \varepsilon\ } FU$$

If we apply U to the above diagram

$$U \xleftarrow{\ U(\varepsilon)\ } UFU$$

and consider now only the images of elements of F,

$$FU \xleftarrow{\ U(\varepsilon_{F(-)})\ } UFUF$$

we get the following picture:

$$\mathrm{id}_{\mathcal{A}} \xrightarrow{\ \eta\ } UF \xleftarrow{\ \mu = U_{\varepsilon_F}\ } UFUF$$

Now, if you write $M = UF$ it is obtained

$$\mathrm{id}_{\mathcal{A}} \xrightarrow{\ \eta\ } M \xleftarrow{\ \mu\ } M \circ M$$

It is beginning to look like the first lecture again ... You have a "unit" η, and a "composition" μ. The miracle is, because of simple things arisen from ε, this operations satisfies the laws of associativity and unity (like monoids, recall the examples of the introductory lecture).

$$
\begin{tikzcd}
M \arrow[r, "\eta_{M(-)}"] \arrow[d, "M(\eta)"'] \arrow[dr, equal, "\mathrm{id}"] & M \circ M \arrow[r, "\mu_M"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\mathrm{assoc}"] & M \circ M \circ M \arrow[d, "M(\mu)"] \\
M \circ M \arrow[r, "\mu"'] & M & M \circ M \arrow[l, "\mu"]
\end{tikzcd}
$$

This is very much like functors from $\mathcal{A}$ to $\mathcal{A}$ and composition $\circ$.

Next time: the reason why η and μ satisfy the conditions.

LECTURE 17

Monads (11/4)

For what follows, compare MacLane, CWM, pp. 78 ff. Consider the diagram

$$
\begin{tikzcd}
\mathcal{X} \arrow[d, shift left, "U"] \\
\mathcal{A} \arrow[u, shift left, "F"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{X}(FA, FUX) \arrow[dr, "\varepsilon_X \circ -"] & \\
\mathcal{A}(A, UX) \arrow[ur, "F_{A,UX}"] \arrow[rr, dotted, shift left, "\varphi_{A,X}"] \arrow[rr, phantom, "\cong"] & & \mathcal{X}(FA, X) \arrow[ll, dotted, shift left, "\psi_{A,X}"] \arrow[dl, "U_{FA,X}"] \\
 & \mathcal{A}(UFA, UX) \arrow[ul, "- \circ \eta_A"] &
\end{tikzcd}
$$

and

(60) $$\varphi_{UX,X}(\mathrm{id}_{UX}) =_{def} \varepsilon_X : FUX \longrightarrow X$$

(61)

$$\psi_{A,FA}(\mathrm{id}_{FA}) =_{def} \eta_A : A \longrightarrow UFA$$

I want to do today Provably the first thing to notice is that:

THEOREM 4. *With φ, psi naturally bijections (natural in A and X as shown), the maps η_X and ε_X are themselves natural in A (or in X) and satisfy:*

(62) $$\varphi_{A,X} = \varepsilon_X \circ F_A, UX(\ )$$

(63) $$\psi_{A,X} = U_{FA,X}(\ ) \circ \eta_A$$

Consequently, it follows that

(64) $$\forall \xi : FA \longrightarrow X,\ \exists! x : A \longrightarrow UX \text{ with } \xi = \varepsilon_X \circ F(x)$$

(65) $$\forall x : A \longrightarrow UX,\ \exists! \xi : FA \longrightarrow X \text{ with } x = \eta_A \circ U(\xi)$$

(66)

Conversely, *Given,* either *natural transformations* ε_X *satisfying (64) or* η_A *satisfying (65), the resulting proposed definitions of* φ *as* $\varepsilon \circ F(-)$ *(or of* ψ *as* $U(-) \circ \eta$*) creates as sort of bijection, natural in* A *and* X *that is exactly meant by the adjointness between* F *and* U.

Once you notice the front and back adjunctions ε_X and η_A (that are really the universal elements Yoneda gives you), (64) and (65) are just the statements about this universality. (Again: we are stating a special case of Yoneda Lemma).

PROOF. Let us verify (62). In order to find that two functors are equal, we just have to evaluate them: Let $x \in \mathcal{A}(A, UX)$ be given, and $\varphi_{A,X}(x) \in \mathcal{X}(FA, X)$.

$$FA \xrightarrow{F(x)} FUX \xrightarrow{\varepsilon_X} X$$

We need to exploit somehow the naturality of φ: For given arrows $A' \xrightarrow{\alpha} A$ and $X \xrightarrow{\beta} X'$ it says that the following diagram commute:

$$\begin{tikzcd}
\mathcal{A}(A, UX) \arrow[r, "\cong", "\varphi_{A,X}"'] \arrow[d, "U(\beta) \circ - \circ \alpha"'] & \mathcal{X}(FA, X) \arrow[d, "\beta \circ - \circ F(\alpha)"] \\
\mathcal{A}(A', UX') \arrow[r, "\cong", "\varphi_{A',X'}"'] & \mathcal{X}(FA', X')
\end{tikzcd}$$

Hence for x we have the equality

$$\varphi(U(\beta) \circ x \circ \alpha) = \beta \circ \varphi(x) \circ F(\alpha)$$

We want to compare

$$\varphi_{A,X}(x) = \varphi_{UX,X}(\mathrm{id}_{UX} \circ x) = \varphi_{UX,X}(\mathrm{id}_{UX}) \circ F(x)$$

See that we are using a tiny part of the naturality that is easy to overlook.

The other part (63) is proved similarly.

Now, the "Conversely". Is just the process of going from, given an element in the ( ) slot in the left hand side of equation (62) and find and element (in the right hand side slot ( )) as the unique solution to (62), or viceversa.

In some sense this theorem is nothing more than a summing up of all we already knew about adjoints. $\square$

From the above discussion, for the diagram

$$\mathrm{id}_{\mathcal{X}} \longrightarrow FU \longrightarrow \mathrm{id}_{\mathcal{X}}$$

we get maps

$$
\begin{tikzcd}
 & FU \arrow[r, "\varepsilon"] & \mathrm{id}_{\mathcal{X}} \\
\mathrm{id}_{\mathcal{X}} \arrow[r, "\eta"] & UF &
\end{tikzcd}
\tag{67}
$$

and applying the first line to objects of type $F(-)$ and applying F to the second line we get

$$
\begin{tikzcd}
FA \arrow[r, "F(\eta_A)"] \arrow[rr, dotted, bend right, "\mathrm{id}_{FA}"'] & FUFU \arrow[r, "\varepsilon_{FA}"] & FA
\end{tikzcd}
$$

This is an instance of equation (62):

$$\varepsilon_X \circ F(\eta_A) = \varphi(\eta_A) = \mathrm{id}_{FA}$$

There is also another way of combining tha maps in (67), and is applying U to the first line, and to objects of the form $U(-)$ in the second line:

$$
\begin{tikzcd}
UX \arrow[r, "F(\eta_{UX})"] \arrow[rr, dotted, bend right, "\mathrm{id}_{UX}"'] & UFUX \arrow[r, "U(\varepsilon_X)"] & UX
\end{tikzcd}
$$

Here we have an instance of equation (63):

$$U(\varepsilon_X) \circ \eta_{UX} = \psi(\varepsilon_X) = \mathrm{id}_{UX}$$

The two unit laws yield (applying F to the top one, and evaluating the bottom one at $A = UX$):

$$
\begin{tikzcd}
 & FUFUX \arrow[dr, "FU_{\varepsilon_X}"] & \\
FUX \arrow[ur, "F(\eta_{UX})"] \arrow[rr, "\mathrm{id}_{FUX}"] \arrow[dr, "F(\eta_{UX})"'] & & FUX \\
 & FUFUX \arrow[ur, "\varepsilon_{FUX}"'] &
\end{tikzcd}
\tag{68}
$$

These equations are *in* the category $\mathcal{X}$. Instead we could have applied U to the bottom, and evaluating the upper diagram at $X = FA$:

(69)
$$\begin{tikzcd}
 & UFUFA \arrow[dr, "U(\varepsilon_{FA})"] & \\
UFA \arrow[ur, "\eta_{UFA}"] \arrow[rr, "\mathrm{id}_{UFA}"] \arrow[dr, "UF(\eta_A)"'] & & UFA \\
 & UFUFA \arrow[ur, "U(\varepsilon_{FA})"'] &
\end{tikzcd}$$

Observe that in this diagram is never an U without and F (UF). So write $T = UF$, keep $\eta = \eta$ and the only place where U appears above without F write[1] $\mu_A = U(\varepsilon_{FA})$. So we have in $\mathcal{A}$:

$$\mu : \mathrm{id}_{\mathcal{A}} \longrightarrow T \qquad \mu_A : T \circ T \longrightarrow T$$

The same token in the other diagram: FU appears everywhere with one exception. So write $G = FU$, keep $\varepsilon = \varepsilon$, and write[2] $\delta = F(\eta_{UX})$. Hence we have in $\mathcal{X}$:

$$G \circ G \xleftarrow{\ \delta\ } G \xrightarrow{\ \varepsilon\ } \mathrm{id}_{\mathcal{X}}$$

So we have a notion of abstract monoid:

(70)
$$\begin{tikzcd}
 & T \circ T \arrow[dr, "\mu"] & \\
T \arrow[ur, "\eta_T"] \arrow[rr, "\mathrm{id}"] \arrow[dr, "T(\eta)"'] & & T \\
 & T \circ T \arrow[ur, "\mu"'] &
\end{tikzcd}$$

The commutativity of this diagram follows from the commutativity of the corresponding diagram (69).

[1]The μ for *m*ultiplication.

[2]δ for *d*iagonal, or co-multiplication.

Similarly, using the commutativity of diagram (68) follows the commutativity of:

$$\begin{tikzcd}
 & G \circ G \arrow[dr, "G_\varepsilon"] & \\
G \arrow[ur, "\delta"] \arrow[rr, "\mathrm{id}"] \arrow[dr, "\delta"'] & & G \\
 & G \circ G \arrow[ur, "\varepsilon_G"'] &
\end{tikzcd}$$

Now, what about the associativity law I predicted?

$$\begin{tikzcd}
T \circ T \circ T \arrow[r, "T(\mu)"] \arrow[d, "\mu_T"'] & T \circ T \arrow[d, "\mu"] \\
T \circ T \arrow[r, "\mu"'] & T
\end{tikzcd} \tag{71}$$

What does it make this work? It is got to be naturality! Let's unravel the diagram (71), by putting again the old meaning of $T = UF$ and $\mu = U(\varepsilon_{FA})$:

$$\begin{tikzcd}
UF \circ UF \circ UF \arrow[r, "UF(\varepsilon_F)"] \arrow[d, "U(\varepsilon_{FUF})"'] & UF \circ UF \arrow[d, "U(\varepsilon_F)"] \\
UF \circ UF \arrow[r, "U(\varepsilon_F)"'] & UF
\end{tikzcd}$$

Now, notice that everything is U of something, so consider the diagram

$$\begin{tikzcd}
FUFUF \arrow[r, "F(\varepsilon_F)"] \arrow[d, "\varepsilon_{FUF}"'] & FUF \arrow[d, "\varepsilon_F"] \\
FUF \arrow[r, "\varepsilon_F"'] & F
\end{tikzcd}$$

and now notice that everyting here is acting on objects of the type $F(-)$. So we get finally the diagram:

$$\begin{tikzcd} FUFU \arrow[r, "F(\varepsilon)"] \arrow[d, "\varepsilon_{FU}"'] & FU \arrow[d, "\varepsilon"] \\ FU \arrow[r, "\varepsilon"'] & \mathrm{id} \end{tikzcd}$$

which commutes because ε is a natural transformation. Hence all the chain above commute because F and U are functors.

The analogous diagram for G

$$\begin{tikzcd} G \circ G \circ G & G \circ G \arrow[l, "G(\delta)"'] \\ G \circ G \arrow[u, "\delta_G"] \arrow[r, "\delta"'] & G \arrow[u, "\delta"'] \end{tikzcd}$$

commutes for essentially the same reasons.

So the first obscure lecture now include the context of categories of functors form $\mathcal{A}$ to $\mathcal{A}$ with compositon $\bullet$ as the composition of functors:

$$M \bullet M \xrightarrow{\mu} M \xleftarrow{\eta} \mathrm{id}_{\mathcal{A}}$$

where μ is the "multiplication" and η is the "unit-selection".

What MacLane would call a *monad* in $\mathcal{A}$ (= monoid for $\circ$ on $\mathcal{A}^{\mathcal{A}}$ is (as Barr pointed out) a *triple* (T, η, μ) with $T \in \mathcal{A}^{\mathcal{A}}$, $\eta : \mathrm{id}_{\mathcal{A}} \longrightarrow T$ and $\mu : T \circ T \longrightarrow T$ such that diagrams (70) and (71)commutes.

LECTURE 18

T-algebras

Definition 11. Given a monad/triple/"standard construction" $(T, \eta, \mu) = \mathbb{T}$ on a category $\mathcal{A}$, by a $\mathbb{T}$-*algebra* is meant a constellation (A, α) with $A \in |\mathcal{A}|$, and $\alpha : TA \longrightarrow A$ for which the following two diagrams commute:

$$\begin{tikzcd} & TA \arrow[dr, "\alpha"] & \\ A \arrow[ur, "\eta_A"] \arrow[rr, "\mathrm{id}_A"'] & & A \end{tikzcd}$$

$$\begin{tikzcd} TTA \arrow[r, "\mu"] \arrow[d, "T\alpha"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd}$$

Examples. (1) Let $A = TB$, and let

$$(\alpha : TA \longrightarrow A) = \mu_B : TTB \longrightarrow TB$$

Then (TB, μ_B) is a $\mathbb{T}$-algebra:

$$\begin{tikzcd} & TTB \arrow[dr, "\mu_B"] & \\ TB \arrow[ur, "\eta_T B"] \arrow[rr, "\mathrm{id}_{TB}"'] & & TB \end{tikzcd}$$

was jsut the half of the unit-law pairs, and

$$\begin{tikzcd}
TTTB \arrow[r, "T\mu_B"] \arrow[d, "\mu_{TB}"'] & TTB \arrow[d, "\mu_B"] \\
TTB \arrow[r, "\mu_B"'] & TB
\end{tikzcd}$$

(2) Other class of examples: Given any adjoint pair of functors

$$\begin{tikzcd}
\mathcal{X} \arrow[r, shift left, "U"] & \mathcal{A} \arrow[l, shift left, "F"]
\end{tikzcd}$$

with form and back adjunctions

$$\eta : \mathrm{id}_{\mathcal{A}} \longrightarrow UF \qquad \varepsilon : FU \longrightarrow \mathrm{id}_{\mathcal{X}}$$

such that $T = UF$, $\eta = \eta$, $U(\varepsilon_F) = \mu$ (in $\mathbb{T} = (T, \eta, \mu)$). In this case $UX, U(\varepsilon_X))$ is a $\mathbb{T}$-algebra for all $X \in |\mathcal{X}|$.

PROOF. In fact, $UX \in |\mathcal{A}|$, $U(\varepsilon_X)$ so $U(\varepsilon_X) : T(UX) \longrightarrow UX$:

$$U\,(FUX \xrightarrow{\varepsilon_X} X) \quad \rightsquigarrow \quad UFUX \xrightarrow{U(\varepsilon_X)} UX$$

Hence, $(A, \alpha) = (UX, U(\varepsilon_X))$ has the right structure to be a candidate. Now

$$\begin{tikzcd}
 & UFUX \arrow[dr, "U(\varepsilon_X)"] & \\
UX \arrow[ur, "\eta_{UX}"] \arrow[rr, "\mathrm{id}_{UX}"'] & & UX
\end{tikzcd}$$

by adjuntion laws. Now, why the following diagram commutes?

$$\begin{tikzcd}
TTUX \arrow[r, "TU\varepsilon_X"] \arrow[d, "\mu_{UX}"'] & TUX \arrow[d, "U(\varepsilon_X)"] \\
TUX \arrow[r, "U(\varepsilon_X)"'] & UX
\end{tikzcd}$$

Because $T = FU$. □

Remark: this is a broader class of examples than that of example (1), and more important: algebras of type (2) are part of a completely natural construction:

(72)
$$\begin{tikzcd}
\mathcal{X} \arrow[rr, "\Phi"] \arrow[dr, shift left, "F"] & & \mathbb{T}\text{-}\mathcal{A}alg \arrow[dl, "U_{\mathbb{T}}"] \\
& \mathcal{A} \arrow[ul, shift left, "U"] &
\end{tikzcd}$$

To endow $\mathbb{T}$-algebras with category structure, define "homomorphism" form (A, α) to (B, β) to mean: any $\mathcal{A}$-morphism $f : A \longrightarrow B$ such that Tf makes the following diagram commute:

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "A"'] & TB \arrow[d, "\beta"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Also, free of charge, we get the preservation of the unit:

$$\begin{tikzcd}
TA \arrow[r, "Tf"] & TB \\
A \arrow[u, "\eta_A"] \arrow[r, "f"'] & B \arrow[u, "\eta_B"']
\end{tikzcd}$$

EXAMPLES.

1. id_A will be homomorphism from (A, α) to itself.
2. If $(A, \alpha) \xrightarrow{f} (B, \beta)$ and $(B, \beta) \xrightarrow{g} (C, \gamma)$ then $g \circ f : (A, \alpha) \longrightarrow (C, \gamma)$. The fact that this composition is a homomorphism follows from the commutativity of the squares in the diagram

$$\begin{tikzcd}
TA \arrow[r, "Tf"'] \arrow[d, "\alpha"'] & TB \arrow[r, "Tg"'] \arrow[d, "\beta"] & TC \arrow[d, "\gamma"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}$$

and the fact that $T(g \circ f) = Tg\, Tf$.

So, $U_{\mathbb{T}}$ is a functor! The only thing left is to check that Φ is a functor. We already dnow how Φ is defined on objects. Let $\xi : X \longrightarrow Y$. For $U_{\mathbb{T}}(\Phi(\xi)) = U(\xi)$ (to commute) it means $\Phi : \Phi(X) \longrightarrow \Phi(Y)$, that is

$$U(\xi) : (UX, \varepsilon_X) \longrightarrow (UY, \varepsilon_Y)$$

We jusst need to check that $U(\xi)$ is $\mathbb{T}$-algebra homomorphism.

$$\begin{tikzcd} TUX \arrow[r, "TU(\xi)"] \arrow[d, "U\varepsilon_X"'] & TUY \arrow[d, "U\varepsilon_Y"] \\ UX \arrow[r, "U(\xi)"'] & UY \end{tikzcd}$$

Writing $T = UF$ this diagram is $U[\ \ ]$ and $[\ \ ]$ does commute! So really Φ becomes a functor.

Now I like to convince you that $U_{\mathbb{T}}$ invariably has an adjoint functor (ΦF will serve ...). To see a left adjoint to $U_{\mathbb{T}} : \mathcal{A}^{\mathbb{T}} \longrightarrow \mathcal{A}$ define

$$F^{\mathbb{T}} : \mathcal{A} \longrightarrow \mathcal{A}^{\mathbb{T}} \qquad F^{\mathbb{T}}(A) = (TA, \mu_A)$$

It is well defined at least at the level of objects, because we checked that this objects are algebras. Now,

$$\begin{tikzcd} A \arrow[d, "g"'] & F^{\mathbb{T}}(A) \arrow[d, "F^{\mathbb{T}}(g)"'] & = & (TA, \mu_A) \arrow[d, "T(g)"] \\ B & F^{\mathbb{T}}(A) & = & (TB, \mu_B) \end{tikzcd}$$

Why $T(g)$ is a $\mathbb{T}$-algebra homomorphism? Let us see:

$$\begin{tikzcd} TTA \arrow[r, "TTg"] \arrow[d, "\mu_A"'] & TTB \arrow[d, "\mu_B"] \\ TA \arrow[r, "Tg"'] & TB \end{tikzcd}$$

Does $F^{\mathbb{T}}(g)$ become a functor? The identity is easy. What about $F^{\mathbb{T}}(g \circ f)$?

Finally, why should $F^{\mathbb{T}}$ (as defined) be a left adjoint to $U_{\mathbb{T}} : \mathcal{A}^{\mathbb{T}} \longrightarrow \mathcal{A}$? We could

$$\mathcal{A}(A, U_{\mathbb{T}}(B,\beta)) \underset{\varphi}{\overset{\psi}{\cong}} \mathcal{A}^{\mathbb{T}}(F^{\mathbb{T}}(A),(B,\beta)) \tag{73}$$

To prove this, it is easier (because of the context) to exhibit the front and back adjoints, i.e. maps $\eta^{\mathbb{T}}$ and $\varepsilon^{\mathbb{T}}$ such that

$$A \xrightarrow{\eta^{\mathbb{T}}_A} U_{\mathbb{T}}F^{\mathbb{T}}A = TA \qquad (TB,\mu_B) = F^{\mathbb{T}}(B) = F^{\mathbb{T}}U^{\mathbb{T}}(B,\beta) \xrightarrow{\varepsilon^{\mathbb{T}}_{(B,\beta)}} (B,\beta)$$

let $\eta^{\mathbb{T}}_A = \eta_A$ and $\varepsilon^{\mathbb{T}}_{(B,\beta)} = \beta$ The naturality of $\eta^{\mathbb{T}}_A$ is reasonable clear. What is not immediately clear is that μ_B are $\mathbb{T}$-algebra homomorphisms.

$$\begin{tikzcd} TTB \arrow[r, "T\beta"] \arrow[d, "\mu_B"'] & TB \arrow[d, "\beta"] \\ TB \arrow[r, "\beta"'] & B \end{tikzcd}$$

Does this commute? This is a particular part of "(B,β) is a $\mathbb{T}$-algebra", exactly expressing β is a $\mathbb{T}$-algebra homomorphism.

Now we know the front and back adjunctions (using the fact that almost everything begin with η and μ it is better to use the two equations below than to go through hom-sets).

(74)

$$\mathrm{id}_{\mathcal{X}} \xrightarrow{\eta^{\mathbb{T}}} U^{\mathbb{T}}F^{\mathbb{T}} \qquad\qquad U^{\mathbb{T}}F^{\mathbb{T}} \xrightarrow{\varepsilon^{\mathbb{T}}} \mathrm{id}_{\mathcal{X}}$$

$$\begin{tikzcd} & U^{\mathbb{T}}F^{\mathbb{T}}U^{\mathbb{T}} \arrow[dr, "U^{\mathbb{T}}\varepsilon^{\mathbb{T}}"] & \\ U^{\mathbb{T}} \arrow[ur, "\eta_{U^{\mathbb{T}}}"] \arrow[rr, "\mathrm{id}?"'] & & U^{\mathbb{T}} \end{tikzcd} \qquad \begin{tikzcd} & F^{\mathbb{T}}U^{\mathbb{T}}F^{\mathbb{T}} \arrow[dr, "\varepsilon^{\mathbb{T}}F^{\mathbb{T}}"] & \\ F^{\mathbb{T}} \arrow[ur, "F^{\mathbb{T}}/\eta_{\mathbb{T}}"] \arrow[rr, "id"'] & & F^{\mathbb{T}} \end{tikzcd}$$

The right-hand diagram commute because $\alpha \circ \eta_A = \mathrm{id}$ is one of the algebra laws. To check the commutativity of the left-hand side diagram, it is enough to apply UF to the whole diagram; we get at the level of objects which clearly commutes:

$$\begin{tikzcd} & TTA \arrow[dr, "\mu_A"] & \\ TA \arrow[ur, "T\eta_A"] \arrow[rr, "\mathrm{id}_A"'] & & TA \end{tikzcd}$$

So the front and back adjoint laws are checked!

Now in (73): To each $\mathcal{A}$-morphism $f : A \longrightarrow B$ (given $\mathbb{T}$-albegra (B, β)) exists a unique $\mathcal{A}^{\mathbb{T}}$-morphism

$$F^{\mathbb{T}}(A) = (TA, \mu_A)\ arr\overset{\hat{f}}{o}w\ (B, \beta)$$

such that $\hat{f} \circ \eta_A = f$. How do you find $\hat{f}$? Indeed $\hat{f}$ is the image of f under the passage

$$\mathcal{A}(A, U_{\mathbb{T}}(B, \beta)) \xrightarrow{F_{\mathbb{T}}} \mathcal{A}^{\mathbb{T}}(F^{\mathbb{T}}A, F^{\mathbb{T}}U^{\mathbb{T}}(B, \beta)) \xrightarrow{\varepsilon_{(B,\beta)} \circ -} \mathcal{A}(F^{\mathbb{T}}A, (B, \beta))$$

in words, $\hat{f} = \varepsilon_{(B,\beta)} \circ F^{\mathbb{T}}(f)$ (this seems completely misterious if not in conjunction with diagram (74) where it comes naturally).

The last thing: were $(T, \eta, \mu) = (UF, \eta, U\varepsilon_F)$ for adjoint pair () then as $F^{\mathbb{T}}$ one could use

$$\Phi \circ F : \mathcal{A} \longrightarrow \mathcal{X} \longrightarrow \mathcal{A}^{\mathbb{T}}$$

LECTURE 19

Proof of Beck's Theorem (11/11)

Let $\mathbb{T} = (T, \eta, \mu)$ a monad, and consider again the diagram

$$
\begin{tikzcd}
\mathcal{X} \arrow[rr, "\Phi"] \arrow[dr, shift left, "U"] & & \mathcal{A}^{\mathbb{T}} \arrow[dl, shift left, "U^{\mathbb{T}}"] \\
& \mathcal{A} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F^{\mathbb{T}}"] &
\end{tikzcd} \tag{75}
$$

When comparing $\Phi \circ F(\cdot)$ and $F^{\mathbb{T}}(\cdot)$

1. Have there the same $U^{\mathbb{T}}$ (each other)?
2. If so, are the α's the same?

Well, given $A \in |\mathcal{A}|$,

$$
\begin{aligned}
\Phi F(A) &= (U^{\mathbb{T}}(\Phi FA), U(\varepsilon_{FA})) \\
&= (UFA, U\varepsilon_{FA}) \\
&= (TA, \mu_A) \\
&= F^{\mathbb{T}}(A)
\end{aligned}
$$

(Is just Φ commute both with U and $F^{\mathbb{T}}$)

Why F is full and faithful on free things?

$$
\begin{tikzcd}
\mathcal{X}(FA, X) \arrow[rr, "\Phi", "\text{locally full \& faithful}"'] \arrow[dr, "\cong"'] & & \mathcal{A}(F^{\mathbb{T}}, \Phi X) \\
& \mathcal{A}(A, UX) \arrow[ur, "\cong"'] &
\end{tikzcd}
$$

Summary: Φ commutes with the U's and with the F's, and each

$$\Phi_{Fa,X} : \mathcal{X}(FA, X) \longrightarrow \mathcal{A}^{\mathbb{T}}(F^{\mathbb{T}}A, \Phi X)$$

is a bijection. *Were* Φ to have a left adjoint (see diagram), $\check{\Phi}$ would have to preserve coequalizers. Now, in $A^{\mathbb{T}}$, for every object $A =$

$(|A|, \alpha)$, the algebra A is actually coequalizer:

$$\begin{tikzcd}
\mathcal{A}^{\mathbb{T}} \arrow[dd, "U^{\mathbb{T}}"'] & F^{\mathbb{T}}(|F^{\mathbb{T}}(|A|)|) \arrow[r, shift left=2, "F^{\mathbb{T}}(\alpha)"] \arrow[r, shift right=2, "\varepsilon_{F^{\mathbb{T}}(|A|)}"'] & F^{\mathbb{T}}(|A|) \arrow[l, dashed, "F(\eta_A)"'] \arrow[r, "\varepsilon_A"] & A \\
 & & & \\
\mathcal{A} & TT(|A|) \arrow[r, shift left=2, "T(\alpha)"] \arrow[r, shift right=2, "T(\eta_{|A|})"'] & T(|A|) \arrow[l, dashed, "\mu_{|A|}"'] \arrow[r, "\alpha"] & {|A|} \\
 & \cdot & \cdot \arrow[l, "\eta_{|A|}"'] & \arrow[l, "\eta_{T|A|}"']
\end{tikzcd} \tag{76}$$

And we know $\eta_{|A|}$ and $\eta_{T|A|}$

$$\begin{aligned}
\alpha\eta_{|A|} &= \mathrm{id} \quad \text{(alg. requirement)} \\
\mu_{|A|}\eta_{T|A|} &= \mathrm{id} \quad \text{(triple requirement)} \\
\mu_{|A|}T(\alpha) &= \mathrm{id}
\end{aligned}$$

The diagonal
by

$$\begin{tikzcd}
{|A|} \arrow[d, "\eta_{|A|}"'] & T(|A|) \arrow[l, "\alpha"'] \arrow[d, "\eta_{T|A|}"] \\
T|A| & TT|A| \arrow[l, "T(\alpha)"]
\end{tikzcd}$$

Also $T(\eta_{|A|})$ works as a "reflexive" map

$$T(\alpha) = T(\eta_A) = T(\alpha \circ \eta_A) = T(\mathrm{id}_{|A|}) = \mathrm{id}_{T|A|}$$

So by an old proposition (see prop. ??) about lifting coequalizers, the algebra A is even is a coequalizer of a $U^{\mathbb{T}}$-split, $U^{\mathbb{T}}$-reflexive pair.

In effect, from coequalizers in $\mathcal{A}^{\mathbb{T}}$ which are $U^{\mathbb{T}}$-split

$$\begin{tikzcd}
E \arrow[r, shift left=2, "x_1"] \arrow[r, shift right=2, "x_2"'] & B \arrow[l, "\delta"] \arrow[r, shift left, "p"] & Q \arrow[l, shift left, "\delta"]
\end{tikzcd}$$

we go to $U^{\mathbb{T}}$-split pairs in $\mathcal{A}^{\mathbb{T}}$

$$\begin{tikzcd}
E \arrow[r, shift left=2, "x_1"] \arrow[r, shift right=2, "x_2"'] & B \arrow[l, "\delta"]
\end{tikzcd}$$

where $E = F^{\mathbb{T}}(?)$ and $B = F^{\mathbb{T}}(??)$. Now, using Φ^{-1} ("replace $F^{\mathbb{T}}$ by F and use fulness of Φ") we get U-split pairs in $\mathcal{X}$

$$\begin{tikzcd} F(?) \arrow[r, shift left=2, "x_1"] \arrow[r, shift right=2, "x_2"'] & F(??) \arrow[l, "\delta"' description] \end{tikzcd}$$

Finally, if U-split pairs of $\mathcal{X}$-morphisms have coequalizers in $\mathcal{X}$ the simple algebra $(|A|, \alpha) = A$ gives rise to a "resolution" (see diagram (76)

$$\begin{tikzcd} F(F^{\mathbb{T}}(|A|)) \arrow[r, shift left, "U\varepsilon_{F(|A|)}"] \arrow[r, shift right, "F(\alpha)"'] & F(|A|) \arrow[r, "p"] & Q \end{tikzcd}$$

Let $\check{\Phi}(|A|, \alpha) = Q$. Now is really a matter of checking that this really does the right job as adjoint: Let $vv \in \mathcal{X}(\check{\Phi}(|A|, \alpha), X)$, and take a test map t and consider the diagrams

$$\begin{tikzcd} FUF(|A|) \arrow[r, shift left, "U\varepsilon_{F(|A|)}"] \arrow[r, shift right, "F(\alpha)"'] & F(|A|) \arrow[r, "p"] \arrow[dr, "t"'] & Q \arrow[d, "v"] \\ & & X \end{tikzcd}$$

$$\begin{tikzcd} UFUF(|A|) \arrow[r, shift left, "\varepsilon_{F(|A|)}"] \arrow[r, shift right, "F^{\mathbb{T}}(\alpha)"'] & F^{\mathbb{T}}(|A|) \arrow[r, "p"] \arrow[dr, "\hat{t}"'] & (|A|, \alpha) \arrow[d] \\ & & \Phi(X) \end{tikzcd}$$

It is not difficult to see that $\hat{t}$ are in 1-1 correspondence with the t. Hence for each v we get a $\hat{v}$ and viceversa.

This works if U-splits pairs have coequalizers in $\mathcal{X}$. Why is this true?

LEMMA 6. *If $U : \mathcal{X} \longrightarrow \mathcal{A}$ reflects coequalizers of U-split pairs, then $\check{\phi} \circ \Phi = \mathrm{id}_{\mathcal{X}}$.*

PROOF.

$$\begin{tikzcd} FUFUX \arrow[r, shift left, "\varepsilon_{F(|A|)}"] \arrow[r, shift right, "F(U\varepsilon_X)"'] & F(|\Phi X|) \arrow[r, "p"] \arrow[dr, "\varepsilon_X"'] & \check{\Phi}\Phi X \arrow[d, "\cong"] \\ & & X \end{tikzcd}$$

□

For $\Phi \circ \check{\phi} = \mathrm{id}_{\mathcal{A}^{\mathbb{T}}}$ one needs, additionally, for U to preserve coequalizers of U-split pairs. So we have a comparison

$$(|A|, \alpha) \longrightarrow \Phi(\check{\Phi}(|A|, \alpha))$$

$\check{\Phi}$ preserves coequalizers. Φ has no good reason to preserve it, but it *has* to preserve those it is forced to. See the diagram

$$\begin{tikzcd}
\cdot \rightrightarrows \cdot \longrightarrow \cdot \arrow[rr, "\Phi"] \arrow[dr, "U"'] & & \Phi(\cdot \rightrightarrows \cdot \longrightarrow \cdot) \arrow[dl, "U^{\mathbb{T}}"] \\
& \mathrm{coeq} &
\end{tikzcd}$$

and recall that $U^{\mathbb{T}}\Phi = U$ and $U^{\mathbb{T}}$ reflects coequalizers of $U^{\mathbb{T}}$-split pairs.

So, Φ preserves coequalizers of U-split pairs if U does. But, the kind of coequalizer $\check{\Phi}(|A|, \alpha)$ constructed had precisely this property.

LECTURE 20

Triples v/s Theories (11/13)

We have the following list of themes to develop:

1. Triples v/s Theories
2. Structure v/s Semantics
3. Rank (when $\mathcal{A} = \mathcal{S}ets$)
4. Birkhoff's Theorem
5. counterexamples

Let us start by the first.

Algebras as I described them at the beginning in terms of sets with operations on one hand, and algebras over a triple on the other seems to have any connection. Here it is:

Fix a category $\mathcal{A}$. Define a "*theory*" over $\mathcal{A}$ to be

1. a functor $\mathcal{A} \xrightarrow{\theta} \Theta$, where $|\Theta| = |\mathcal{A}|$ and $\theta(A) = A$, and
2. another functor $\Theta \xrightarrow{\tilde{\theta}} \mathcal{A}$, right adjoint to θ (with fixed choice of front and back adjunctions).

By a category of θ-algebras mean the pullback in $\mathcal{C}at$:

(77)
$$\begin{tikzcd}
\Theta\text{-}\mathcal{A}lg \arrow[r, dashed] \arrow[d, dashed] & \mathcal{S}ets^{\Theta^{op}} \arrow[d, "-\circ\theta"] \\
\mathcal{A} \arrow[r, "Y"'] & \mathcal{S}ets^{\mathcal{A}^{op}}
\end{tikzcd}$$

(The same as before, but instead of $\mathcal{S}ets$ we have an arbitrary category $\mathcal{A}$). The expanded definition of $-\circ\theta$ follows from

(78)
$$\mathcal{A}^{op} \xrightarrow{\theta} \Theta^{op} \xrightarrow{\mathrm{a}} \mathcal{S}ets$$

Also, recall that an element of $\mathcal{S}ets^{\Theta^{op}}$ is a pair (A, a), where $\mathrm{a} : \Theta^{op} \longrightarrow \mathcal{S}ets$, and $\mathrm{a}(n) = \mathrm{a}(\theta(n)) = \mathcal{A}(n, A)$. ($\mathcal{A}(n, A)$ "smells" like A^n –which is the case in $\mathcal{S}ets$, or in other words, "is what $\mathcal{A}$ thinks is an n-tuple in A")

Hence we have the diagram:

$$\begin{tikzcd}[row sep=small]
\text{in } \Theta & & & \text{(in } \mathcal{S}ets\text{)} \\
n & \mathrm{a}(n) \arrow[d, "\mathrm{a}(\alpha)"] & = \mathcal{A}(n,A) \arrow[d, "\mathrm{a}(\alpha)"] & A^n \arrow[d, "\mathrm{a}(\alpha)"] \\
k \arrow[u, "\alpha"] & \mathrm{a}(k) & = \mathcal{A}(k,A) & A^k
\end{tikzcd}$$

For $f : n \longrightarrow A$ is really worth rewriting it

$$\{\mathrm{a}(\alpha)\}(f) = f * \alpha : \ k \longrightarrow A$$

This comes from

$$\begin{tikzcd}
l \arrow[r, "\beta", "\Theta\text{-mor.}"'] \arrow[rr, dashed, bend right, "\alpha \circ \beta"'] & k \arrow[r, "\alpha", "\Theta\text{-mor.}"'] \arrow[rr, dashed, bend left, "f \circ \alpha"] & n \arrow[r, "f", "\mathcal{A}\text{-mor.}"'] & A
\end{tikzcd}$$

And with this it is easy to get the following equations:

Preserve composition means:

$$(f * \alpha) * \beta = f * (\alpha \circ \beta)$$

Preserve identity means:

$$f * \mathrm{id}_k = f$$

Finally, for $a \in \mathcal{A}(k, n)$, from (78),

$$f * \theta(a) = f \circ a$$

Notice that the formalism is exactly the same as in the case of $\mathcal{S}ets$.

In (77) we can name U_θ and we can construct a functor F_θ : $\mathcal{A} \longrightarrow \Theta\text{-}\mathcal{A}lg$. How? using the monad that is lying behind the diagram. Look back to θ and $\tilde{\theta}$ in (1), (2) of the definition of theory: we have already a monad on $\mathcal{A}$. So write

$$\mathbb{T} = (T, \eta, \mu) \quad \text{where } T = \tilde{\theta} \circ \theta$$

and where η and μ "are" the front and back adjunctions for $(\theta, \tilde{\theta})$. Consider the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[dr, dotted, "F_\theta"] \arrow[drr, "A \mapsto \Theta(-,\theta(A))"] \arrow[ddr, "T"'] & & \\
& \Theta\text{-}\mathcal{A}lg \arrow[r] \arrow[d, "U_\theta"] & \mathcal{S}ets^{\Theta^{op}} \arrow[d, "- \circ \theta"] \\
& \mathcal{A} \arrow[r, "Y"'] & \mathcal{S}ets^{\mathcal{A}^{op}}
\end{tikzcd}$$

and check that

$$\Theta(\theta(-), \theta(A)) = \mathcal{A}(-, \tilde{\theta}\theta(A)) = \mathcal{A}(-, TA)$$

Hence, the diagram above (because Θ-$\mathcal{A}lg$ is a pullback gives an arrow F_θ). So we have a functor F_θ running upwards in (77) We can prove (in the same way as we did in the case of $\mathcal{S}ets$) that F_θ is an adjoint to U_θ (the generalization is very straightforward).

Now, fix a monad $\mathbb{T} = (T, \eta, \mu)$. On the one hand, there is $\mathcal{A}^{\mathbb{T}}$, a $\mathbb{T}$-algebra. On the other, there is the $\Theta_{\mathbb{T}}$ with

$$\Theta_{\mathbb{T}}(k, n) = \begin{cases} 1. \ \mathcal{A}(k, Tn) \\ 2. \ \mathcal{A}^{\mathbb{T}}(F^{\mathbb{T}}k, F^{\mathbb{T}}n) \\ 3. \ \mathrm{nat}((U^{\mathbb{T}})^n, (U^{\mathbb{T}})^k) \end{cases}$$

where

$$U^{\mathbb{T}} : \mathcal{A}^{\mathbb{T}} \longrightarrow \mathcal{A} \qquad (U^{\mathbb{T}})^n = \mathcal{A}(n, U^{\mathbb{T}}(-)) : \mathcal{A}^{\mathbb{T}} \longrightarrow \mathcal{S}ets$$

It is easy to see that these three things are interchangeable (see the same proof for the case of $\mathcal{S}ets$ in lecture 14).

Consider the functor

$$\theta : \mathcal{A} \longrightarrow \Theta_{\mathbb{T}}$$

defined on objects as $\theta(A) = A$, and on arrows as

$$\theta(k \xrightarrow{f} n = - \circ f = (U^{\mathbb{T}})^f$$

Now, what can we use for $\tilde{\theta}$? Define simply

$$\tilde{\theta} : \Theta_{\mathbb{T}} \longrightarrow \mathcal{A}$$

acting on objects as $\tilde{\theta}(A) = TA$ and on morphism as the corresponding underlying morphism.

The monad arising from this adjoint is the same as the original.

Why algebras from the point of view of monads?. Why, given $\mathbb{T}$, should $\mathcal{A}^{\mathbb{T}}$ and $\Theta_{\mathbb{T}}$-$\mathcal{A}lg$ the same? Because

First point of view. Each has an underlying functor

$$\begin{tikzcd} \arrow[r, shift left, "U"] & \mathcal{A} \arrow[l, shift left, "F"] \end{tikzcd}$$

with left adjoint F and "composition-triple" just $\mathbb{T}$ again. Each fulfills the hypothesis of Beck's Theorem, and each looks like $\mathcal{A}^{\mathbb{T}}$.

Second point of view. (Or how this things connect explicitly).

Let's look at

$$\begin{tikzcd}
\mathcal{A}^{\mathbb{T}} \arrow[dr, dotted] \arrow[drr, "{(A,\alpha) \mapsto \mathcal{A}^{\mathbb{T}}(-,(A,\alpha))}"] \arrow[ddr, "U^{\mathbb{T}}"'] & & \\
& \Theta_{\mathbb{T}}\text{-}\mathcal{A}lg \arrow[r] \arrow[d, "U_{\Theta_{\mathbb{T}}}"] & \mathcal{S}ets^{\Theta^{op}} \arrow[d, "F^{\mathbb{T}} \circ -"] \\
& \mathcal{A} \arrow[r, "Y"'] & \mathcal{S}ets^{\mathcal{A}^{op}}
\end{tikzcd}$$

An element $(A, \alpha) \in \mathcal{A}^{\mathbb{T}}$ (upper-left corner) "going through" the upper-right corner gives the path:

$$(A, \alpha) \rightsquigarrow \mathcal{A}^{\mathbb{T}}(-, (A, \alpha)) \rightsquigarrow \mathcal{A}^{\mathbb{T}}(F^{\mathbb{T}}(-), (A, \alpha))$$

and "going through" the left-lower corner:

$$(A, \alpha) \rightsquigarrow U^{\mathbb{T}}(A, \alpha) \rightsquigarrow \mathcal{A}(-, U^{\mathbb{T}}(A, \alpha))$$

Now, because $F^{\mathbb{T}}$ and $U^{\mathbb{T}}$ are adjoints, the final results are $\cong$ (moreover, are equal choosing the right ...).

So (because $\Theta_{\mathbb{T}}$-$\mathcal{A}lg$ is a pullback) we have a map $\mathcal{A}^{\mathbb{T}} \longrightarrow \Theta_{\mathbb{T}}$-$\mathcal{A}lg$. What does it do? Let $k \xrightarrow{\omega} n$ in $\Theta_{\mathbb{T}}$ and $n \xrightarrow{f} A$ in $\mathcal{A}$. Notice that ω

also $k \xrightarrow{\omega} Tn$. Hence define $f * \omega$ (in $\mathcal{A}$) by

$$\begin{tikzcd} Tn \arrow[r, "Tf"] & TA \arrow[d, "\alpha"] \\ k \arrow[u, "\omega"] \arrow[r, "f * \omega"'] & A \end{tikzcd}$$

So, we get a map

$$(A, \alpha) \mapsto (A, \mathrm{a})$$

Now, how do we get from $\Theta_{\mathbb{T}}\text{-}\mathcal{A}lg \longrightarrow \mathcal{A}^{\mathbb{T}}$?
Fix $(A, \mathrm{a}) \in \Theta_{\mathbb{T}}\text{-}\mathcal{A}lg$ and define

$$(A, \mathrm{a}) \mapsto (A, U(\varepsilon_{(A,\mathrm{a})})$$

Now check that this is really in $\mathcal{A}^{\mathbb{T}}$ (α satisfy the conditions of $\mathbb{T}$-algebras).

(Complete this part!)

LECTURE 21

Structure v/s Semantics I: The semantics functor. (11/18)

Think of this words more as a slogan than technical words. The v/s here –unlike in the case of triples v/s theories– is refering to an adjoint pair of contravariant functors.

$$\text{adj. cat. over } \mathcal{A} \qquad\qquad\qquad \text{Monads } (\mathcal{A})$$

(79)

$$\begin{tikzcd}
\mathcal{X} \arrow[d, shift left, "U"] & \\
\mathcal{A} \arrow[u, shift left, "F"] \arrow[r, phantom, ""] & \\
\end{tikzcd}
\quad \xdashrightarrow{\text{Structure}} \quad (UF, \eta, U_{\varepsilon_F})$$

$$\begin{tikzcd}
\mathcal{A}^{\mathbb{T}} \arrow[d, shift left, "U^{\mathbb{T}}"] \\
\mathcal{A} \arrow[u, shift left, "F^{\mathbb{T}}"]
\end{tikzcd}
\quad \xdashleftarrow[\text{Semantics}]{} \quad \mathbb{T} = (T, \eta, \mu)$$

So, let's back up a little bit. What does it mean a pair of contravariant functors? Consider the following familiar example:

$$\begin{array}{ccc} \mathcal{KT}_2 & \xrightarrow{C} & \text{Real top. vect. spaces} \\ X & \mapsto & C(X) \end{array}$$

where $\mathcal{KT}_2$ is the category of compact hausdorff spaces and C is $C(X)$ is the set of continuous real-valued function on X. We have

$$C(X) = \mathcal{T}op(X, \mathbb{R}) = \mathcal{KT}op$$

What could come back playing the role of an adjoint? Let's see:

$$f : L \longrightarrow \mathrm{Cont}(X, \mathbb{R})$$

can be thought as

$$f : L \times X \longrightarrow \mathbb{R}$$

such that each $f(l, =)$ is continuous in $=$, and each $f(-, x)$ is linear in $-$. Hence,

$$\mathrm{Cont}(X, \mathbb{R}^L) \cong \mathrm{Cont}(X, \mathrm{Lin}(L, \mathbb{R}))$$

So we have a kind of duality between

$$(L, \mathrm{Cont}(X, \mathbb{R})) \quad \text{and} \quad (X, \mathrm{Lin}(L, \mathbb{R}))$$

(think of $\mathrm{Cont}(X, \mathbb{R})$ as $F(L)$, and of $\mathrm{Lin}(L, \mathbb{R})$ as $U(X)$)

So we have a pair of contravariant functors D and Δ[1]

$$\begin{tikzcd}
\mathcal{A}^{op} \arrow[r, shift left, "\mathrm{id}"] \arrow[rr, bend left, dashed, "D^{op}"] & \mathcal{A} \arrow[l, shift left] \arrow[r, shift left, "D"] & \mathcal{B} \arrow[l, shift left, "\Delta"] \arrow[ll, bend left, dashed, "\Delta_{op}"]
\end{tikzcd}$$

are *mutually adjoint on the right* if

$$\mathcal{B}(B, D^{op}(A)) \cong_{B,A} \mathcal{A}(A, \Delta(B)) \cong_{B,A} \mathcal{A}^{op}(\Delta_{op}(B), A)$$

Δ_{op} is left adjoint to D^{op} (also D_{op} is left adjoint to Δ^{op}).

This is what is going to happen in diagram (79).

structure v/s semantics. Fix $\mathcal{A}$ an consider elements of $\mathcal{A}$ as categories $\mathcal{X}$ equiped with a $\mathcal{A}$-functor U that comes with a left adjoint and maps X

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "X"] \arrow[d, shift left, "U"] & \mathcal{X}' \arrow[d, shift left, "U'"] \\
\mathcal{A} \arrow[u, shift left, "F"] \arrow[r, phantom, "="] & \mathcal{A} \arrow[u, shift left, "F'"]
\end{tikzcd}$$

Call it *adjointed categories over* $\mathcal{A}$. The name *Structure* is for (monadic) structures, the only important categorical aspect of the adjoint pair

$$\begin{tikzcd}
\mathcal{X} \arrow[r, shift left, "U"] & \mathcal{A} \arrow[l, shift left, "F"]
\end{tikzcd}$$

Why the other is called *Semantics*? It has remained a mistery for me for more than 30 years ... (It has logical connotations, as the "concrete" object that is a "realization" of a theory and has this sense in Lawvere).

Is there a obvious way to make $\mathrm{Monad}(\mathcal{A})$ a category? As maps from one triple $\mathbb{T} = (T, \eta, \mu)$ to another $\mathbb{T}' = (T', \eta', \mu')$ (both in the same category $\mathcal{A}$) use any natural transformation $\lambda : T \longrightarrow T'$ for which

[1]Obviously the d's are for dual, the natural examples of this kind of functors.

(Recall the analogy with the category $\mathcal{A}^{\mathcal{A}}$, composition $\circ$ and identity $\mathrm{id}_{\mathcal{A}}$):

$$\begin{tikzcd}
T \arrow[rr, "\lambda"] & & T' \\
& \mathrm{id}_{\mathcal{A}} \arrow[ul, "\eta"] \arrow[ur, "\eta'"'] &
\end{tikzcd}$$

λ converts the unit for T into the unit for T'. For the composition,

$$\begin{tikzcd}
T \circ T \arrow[r, "\lambda \circ \lambda"] \arrow[d, "\mu"'] & T' \circ T' \arrow[d, "\mu'"] \\
T \arrow[r, "\lambda"'] & T
\end{tikzcd}$$

where $\lambda \circ \lambda$ is the common value of the outside legs of the following diagram (which commutes because λ is a natural transformation):

$$\begin{tikzcd}
& T' \circ T \arrow[dr, "T'(\lambda)"] & \\
T \circ T \arrow[ur, "\lambda_T"] \arrow[rr, dashed, "\lambda \circ \lambda"] \arrow[dr, "T(\lambda)"'] & & T' \circ T' \\
& T \circ T' \arrow[ur, "\lambda_{T'}"'] &
\end{tikzcd}$$

This is exactly the same thing that happened for monoids and sets in your first steps in algebra: given a function $h : M \longrightarrow M'$, get a map from $M \times M \longrightarrow M' \times M'$:

$$\begin{tikzcd}
M \times M \arrow[r, "h \times h"] \arrow[d] & M' \times M' \arrow[d] \\
M \arrow[r, "h"'] & M'
\end{tikzcd}
\qquad
\begin{tikzcd}
& M' \circ M \arrow[dr, "id_{M'} \times h"] & \\
M \times M \arrow[ur, "h \times \mathrm{id}_M"] \arrow[rr, dashed, "h \times h"] \arrow[dr, "\mathrm{id}_M \times h"'] & & M' \times M' \\
& M \times M' \arrow[ur, "h \times \mathrm{id}_{M'}"'] &
\end{tikzcd}$$

Exercise (extremely straightforward) With this definitions of objects and maps, Monad($\mathcal{A}$) becomes a category.

Now, how are Structure and Semantics functors? We are going to see that they are adjoints. Consider the diagram

$$aa$$

Let us take two monads and a transformation as in

$$\mathbb{T} = (T, \eta, \mu) \xrightarrow{\lambda} \mathbb{T}' = (T', \eta', \mu')$$

For each A, given α' we get $\alpha = \alpha' \circ \lambda_A$ (see the diagram below):

(80)
$$\begin{tikzcd} TA \arrow[rr, "\lambda_A"] \arrow[dr, "\alpha"'] & & T'A \arrow[dl, "\alpha'"] \\ & A & \end{tikzcd}$$

This gives you an idea of how the contravariant functor is behaving. If (A, α') really is a $\mathbb{T}'$-algebra, (why) is $(A, \alpha' \circ \lambda_A)$ a $\mathbb{T}$-algebra? We need to see whether (and why) the following diagram commutes:

(81)
$$\begin{tikzcd} & TA \arrow[d, "\lambda_A"] \arrow[ddr, "\alpha"] & \\ & T'A \arrow[dr, "\alpha'"] & \\ A \arrow[uur, "\eta'_A"] \arrow[ur, "\eta'"] \arrow[rr, "\mathrm{id}_A"] & & A \end{tikzcd}$$

We can see that the upper right triangle commutes because ..., the upper left by definition of α, and the one in the bottom because $\mathbb{T}'$ is an algebra. Similarly, using the hypothesis about the maps that appear in it, you can check the commutativity of the following square:

$$
\begin{tikzcd}
TTA \arrow[rrr, "T\alpha"] \arrow[dr, "T\lambda_A"] \arrow[ddr, "\lambda \circ \lambda"'] \arrow[dddd, "\mu_A"'] & & & TA \arrow[dll, "T\alpha'"'] \arrow[ddl, "\lambda_A"] \arrow[dddd, "\alpha"] \\
& T(T'A) \arrow[d, "\lambda_{T'A}"] & & \\
& T'T'A \arrow[r, "\mu'_A"'] \arrow[d, "\mu_{A'}"] & T'A \arrow[ddr, "\alpha'"] & \\
& T'A \arrow[drr, "\alpha'"] & & \\
TA \arrow[ur, "\lambda_A"'] \arrow[rrr, "\alpha"'] & & & A
\end{tikzcd}
$$

Replacing legs by equivalent legs you reach the desired goal. This complete the proof that taking a (A, α),

$$
\begin{tikzcd}
\mathcal{A}^{\mathbb{T}'} \arrow[rr, "- \circ \lambda"] \arrow[dr, "U^{\mathbb{T}'}"'] & & \mathcal{A}^{\mathbb{T}} \arrow[dl, "U^{\mathbb{T}}"] \\
& \mathcal{A} &
\end{tikzcd}
$$

Now it is just a matter of checking whether the algebra homomorphism condition persists: if (A, α') and (B, β') are two $\mathbb{T}'$-algebras and $f : A \longrightarrow B$ is a $\mathbb{T}'$-homomorphism from $(A, \alpha' \circ \lambda)$ to $(B, \beta' \circ \lambda_B)$.

EXAMPLE. Let R be a ring, M an R-module. Consider

$$R \otimes M \longrightarrow M, \qquad R \longrightarrow R$$

We have

$$
\begin{tikzcd}
R \otimes M \arrow[r] \arrow[dr] & R' \otimes M \arrow[d] \\
& M
\end{tikzcd}
$$

This is completely the analogous, and even the proof is analogous.

Writing the equations of this will give a completely uninteligible proof, but a diagram shows it immediately:

$$\begin{tikzcd}
TA \arrow[rrr, "Tf"] \arrow[dd, "\alpha"'] \arrow[dr, "\lambda_A"] & & & TB \arrow[dd, "\beta"] \arrow[dl, "\lambda_B"] \\
& T'A \arrow[r, "T'f"] \arrow[dl, "\alpha'"] & T'B \arrow[dr, "\beta'"] & \\
A \arrow[rrr, "f"] & & & B
\end{tikzcd}$$

The upper square commutes because λ is a natural transformation, and the bottom one because f is a T' homomorphism. The triangles on the sides commute because of diagram (80) above.

This does the story of the semantic functor. Now, why structure is a functor? Next lecture.

LECTURE 22

Stucture v/s Semantics II: The structure functor (11/20)

In last lecture we saw how Structure behaved on objects. Now I want to show that it behaves as functor on 'maps' as well. Let us consider the diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, "X"] \arrow[dr, shift left, "U"] & & \mathcal{X}' \arrow[dl, shift left, "U'"'] \\
& \mathcal{A} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F'"'] &
\end{tikzcd}$$

We have $U' \circ X = U$. Also we have two triples on $\mathcal{A}$: $\mathbb{T} = (UF, \eta, U_{\varepsilon_F})$ and $\mathbb{T}' = (U'F', \eta', U_{\varepsilon_{F'}})$ coming from the adjoints pair F, U and F', U' respectively.

Claim: the equation $U' \circ X = U$ will drive us into recognizing a way of going from $\mathbb{T}$ to $\mathbb{T}'$.

First, we ...to go from $U'F'A \longrightarrow UFA$.

$$\lambda_A : U'F'A \xrightarrow{U'(F'A \longrightarrow X(FA))} U'(X(FA)) = UFA$$

In $\mathcal{X}'$ we have

$$\begin{aligned}
\mathcal{X}'(F'A, X(?)) &\cong \mathcal{A}(A, U'(X(?))) \quad ? \in |\mathcal{X}| \\
&= \mathcal{A}(A, U(?)) \\
&= \mathcal{X}(FA, ?)
\end{aligned}$$

Now, when $? = FA$ we can ask: what lands on the identity of FA?

Let Λ_A be the precursor of id_A and let $\lambda_A = U'(\Lambda_A)$. Now, id_A is natural in A, hence λ is natural.

Exercise. Prove the η identities. Hint: follow from the fact that $\Lambda_A : F'(A) \longrightarrow X(FA)$ was an $\mathcal{X}'$-morphism

The compatibility comes from a different source: In the equations above, both front adjunctions are build in the process

$$U'\Lambda_A \circ \eta_A = \lambda_A \circ \eta'_A = (\eta_A)$$

In particular

$$\begin{tikzcd}
\mathcal{A}^{\mathbb{T}} \arrow[r, "X"] & \mathcal{A}^{\mathbb{T}'} \\
\Theta_{\mathbb{T}} \arrow[u] & \Theta_{\mathbb{T}'} \arrow[u] \arrow[l, dotted] \\
\mathcal{A} \arrow[u, "\theta"'] \arrow[r, phantom, "="] & \mathcal{A} \arrow[u]
\end{tikzcd}$$

Is there a natural way to walk the path $\cdots$? Yes. One way to interpret equations above in the case $? = F'(\ )$ (also we are writing F' instead of $F^{\mathbb{T}'}$ and F for $F^{\mathbb{T}}$):

$$\begin{aligned}
\Theta_{\mathbb{T}'}(A, B) &\cong \mathcal{A}^{\mathbb{T}'}(F'A, F'B) \\
&\cong \mathcal{A}^{\mathbb{T}'}(F'A, X(FB)) \\
&\cong \mathcal{A}(A, U'X(FB)) \\
&\cong \mathcal{A}(A, U(FB)) \\
&\cong \mathcal{A}^{\mathbb{T}}(FA, FB) \\
&\cong \Theta_{\mathbb{T}}(A, B)
\end{aligned}$$

We can go from the "free" objects in $\mathcal{A}^{\mathbb{T}}$ to $\mathcal{A}^{\mathbb{T}}$ through this path:

$$\begin{aligned}
\mathcal{A}^{\mathbb{T}'}(F'A, X(?)) &\cong \mathcal{A}(A, U'X(?)) \\
&\cong \mathcal{A}(A, U(?)) \\
&\cong \mathcal{A}^{\mathbb{T}}(FA,\ ?) \\
&\cong \mathcal{A}^{\mathbb{T}}(\check{X}(F'A),\ ?)
\end{aligned}$$

But we saw this situation before. How can we extend this proposition to all of $\mathcal{A}^{\mathbb{T}}$? Just the same as before (recall that $\mathcal{A}^{\mathbb{T}}$ has the equalizers we need for that construction).

Remember, from the perspective of theories, algebras were a pullback:

$$\begin{tikzcd} \Theta\text{-}\mathcal{A}lg \arrow[r] \arrow[d] & \mathcal{S}ets^{\Theta^{op}} \arrow[d, "\mathcal{A} \xrightarrow{\theta} \Theta"] \\ \mathcal{A} \arrow[r, "Y"'] & \mathcal{S}ets^{\mathcal{A}^{op}} \end{tikzcd}$$

On the other hand:

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F"] & \mathcal{X} \arrow[l, shift left, "U"] \end{tikzcd}$$

declaring

$$\Theta_U(A,B) = \mathrm{nat}(U^B, U^A) = \mathrm{nat}(\mathcal{A}(B, U(-)), \mathcal{A}(A, U(-))$$

The claim is somehow that the passage from such a scheme Theory$/\mathcal{A}$ to a $\mathcal{C}at/\mathcal{A}$.

If we look at $\mathrm{nat}(U^B, U^A)$ it becomes direct to recognize that

$$\begin{tikzcd}[column sep=small] & \Theta_U(A,B) = \mathrm{nat}(U^B, U^A) = \mathrm{nat}(\mathcal{A}(B, U'(X(-))), \mathcal{A}(A, U'(X(-)))) \\ \mathcal{A} \arrow[ur, "\theta"] \arrow[dr, "\theta'"'] & \\ & \Theta_{U'}(A,B) = \mathrm{nat}((U')^B, (U')^A) = \mathrm{nat}(\mathcal{A}(B, U'(-)), \mathcal{A}(A, U'(-))) \end{tikzcd}$$

This is how structure from the theories perspective is even easier than the monad perspective.

Semantics from the .....perspective is easier, because ....

Checking the adjointness (we haven't done it for monad perspective yet)

$$\mathrm{Th}(\Theta, \mathrm{Str}(\mathcal{X}' \longrightarrow \mathcal{A}))$$

$$\mathcal{C}at/\mathcal{A}(\mathcal{X}' \longrightarrow \mathcal{A},\ \Theta\text{-}\mathcal{A}lg \longrightarrow \mathcal{A})$$

$$\begin{tikzcd}
\mathcal{X}' \arrow[dr, dotted] \arrow[drr] \arrow[ddr] & & \\
& \Theta\text{-}\mathcal{A}lg \arrow[r] \arrow[d] & \mathcal{S}ets^{\Theta^{op}} \arrow[d] \\
& \mathcal{A} \arrow[r] & \cdot
\end{tikzcd}$$

The arrow from $\mathcal{X}'$ to $\mathcal{A}$ is already given. The one from $\mathcal{X}'$ to $\mathcal{S}ets^{\Theta^{op}}$ is got as follows:

$$\mathcal{X}' \longrightarrow \mathcal{S}ets^{\Theta^{op}}$$

$$\mathcal{X}' \times \Theta^{op} \longrightarrow \mathcal{S}ets$$

$$\Theta^{op} \longrightarrow \mathcal{S}ets^{\mathcal{X}'}$$

$$\Theta \longrightarrow (\mathcal{S}ets^{\mathcal{X}'})^{op}$$

Now, the last line factors

$$\Theta \longrightarrow \Theta_{U'} \longrightarrow (\mathcal{S}ets^{\mathcal{X}'})^{op}$$

where the first arrow comes from the condition of the pullback in (**??**).

From that perspective structure and semantics:

inverse element $\rightsquigarrow$ monad level

composition of functors $\rightsquigarrow$ theories level.

LECTURE 23

Birkhoff's Theorem (11/25)

See Manes, pp. ....

In this lecture we are going to talk about Birkhoff's Theorem (also called HSP theorem) and how it fits in our framework.

In its original form it said:

THEOREM 5 (Birkhoff). *If* (Δ, n) *is a* finitary *(that is, each set* $n(\omega)$ *is finite for all* $\omega \in \Delta$*) "signature" and* $\mathcal{V} \subseteq \Delta\text{-}\mathcal{A}lg$ *is a "class of algebras", then* $\mathcal{V}$ *is closed under the formation of products, subalgebras and homomorphic images if and only if* $\mathcal{V} = (\Delta, \varepsilon)\text{-}\mathcal{A}lg$ *for some system of equations* ε *on* $\Delta\text{-}\mathcal{A}lg$.

The next stage is due to Słomiński: Same "then", different "if": signature not necessarily finitary, but just "bounded" ($\exists \aleph : \forall \omega \in \Delta, n(\omega) < \aleph$). Both, Birkhoff's and Słomiński's theorem are formulated so: in terms of some preexisting operations, what does it take to be ....

Lawvere's and Beck's theorems are different in that they don't focus in a certain category of algebras, but in an arbitrary category $\mathcal{A}$ with a functor $U : \mathcal{A} \longrightarrow \mathcal{S}ets$.

How do they fit in our general setting? In the process of going from the closure properties to the definition of an equational class, one needs and adjoint functor $\mathcal{V} \leftarrow \Delta$. Once one translate free algebras to free algebras in $\mathcal{V}$ you are ready.

Back to the pages of Manes.

DEFINITION 12. Given a monad π in a category $\mathcal{A}$, a subcategory

$$\mathcal{V} \xrightarrow{i} \mathcal{A}^{\mathbb{T}}$$

is called an "abstract Birkhoff category" if

(B1) For all $F^{\mathbb{T}}(A)$ there is $F(A) \in |\mathcal{V}|$ such that

$$\mathcal{V}(F(A), V) \cong \mathcal{A}^{\mathbb{T}}(F^{\mathbb{T}}(A), i(V)) \tag{82}$$

is natural in V.

(B2) For all $V \in |\mathcal{V}|$, for all $(A, \alpha) \in \mathcal{A}^{\mathbb{T}}$, for all $f : i(V) \longrightarrow (A, \alpha)$ with $U^{\mathbb{T}}(f)$ split epi, then

$$(A, \alpha) = i(? \in |\mathcal{V}|) \text{ i.e. ``}(A, \alpha) \in |\mathcal{V}|\text{''}$$

(B3) $\mathcal{V}$ is meant to be *full* subcategory of $\mathcal{A}^{\mathbb{T}}$.

(B2) tries to "translate" the homomorphic images in the best way it can be expressed here, close under the formation of quotients. $U^{\mathbb{T}}(f)$*split epi* means that there exists $g \in \mathcal{A}(A, U^{\mathbb{T}}V)$ with $U^{\mathbb{T}} \circ g = \mathrm{id}_A$.

First thing to observe:

(B4) the functor $i : \mathcal{V} \longrightarrow \mathcal{A}^{\mathbb{T}}$ has a full-fledged left adjoint *and*

(B5) $\mathcal{V}$ has coequalizers (same as in $\mathcal{A}^{\mathbb{T}}$) ofr U-split pairs ($U =_{def} U^{\mathbb{T}} \circ i$).

The reason for this: if you had (82), you get a left adjoint in the following way:

$$\mathcal{V}(F(A), V) \cong \mathcal{A}^{\mathbb{T}}(F^{\mathbb{T}}(A), i(V)) \cong \mathcal{A}(A, U(V))$$

Now if you know where to send the free algebra in a way that is consistent with

$$\begin{tikzcd} \mathcal{K} \arrow[r, shift left, "i"] & \mathcal{A}^{\mathbb{T}} \arrow[l, shift left, "\hat{i}"] \end{tikzcd}$$

$$\mathcal{K}(\hat{i}(F^{\mathbb{T}}(A)), K) \cong \mathcal{A}^{\mathbb{T}}(F^{\mathbb{T}}(A), iK)$$

In such a situation you can extend to arbitrary algebras if $\mathcal{K}$ has enough coequalizers

$$\begin{tikzcd} F^{\mathbb{T}}F^{\mathbb{T}}(A) \arrow[r, shift left, "F^{\mathbb{T}}(\alpha)"] \arrow[r, shift right, "\mu"'] & F^{\mathbb{T}}(A) \arrow[r, "\varepsilon_A", "\alpha"'] & (A, \alpha) \end{tikzcd}$$

Exercise: prove this. Hint:

$$\begin{tikzcd}[column sep=small] & & \mathcal{V} & & & & \mathcal{A}^{\mathbb{T}} & \\ \circ \arrow[r, shift left] \arrow[r, shift right] & \circ \arrow[r, "`f'"] & \bullet \arrow[rr, dotted, "i"] \arrow[dr, dotted] & & \circ \arrow[r, shift left] \arrow[r, shift right] & \circ \arrow[r, "f"] \arrow[dl, dotted, "U^{\mathbb{T}}"] & (A, \alpha) & \\ & & & \circ \arrow[r, shift left] \arrow[r, shift right] & \circ \arrow[r] & {} & & \end{tikzcd}$$

(B2) says in this case: there is $\bullet$ such that $V \xrightarrow{`f'} \bullet\circ \quad \rightsquigarrow \xrightarrow{f} (A, \alpha)$
Using the property that i doesn't colapse any map (is the inclusion), it follows that $\circ \rightrightarrows \circ \xrightarrow{`f'} \bullet$ is a coequalizer.

(B5) is a result of (B2). So (B4) is (B1) and (B5) combined. Next

(B6) $U : \mathcal{V} \longrightarrow \mathcal{A}$ is monadic (*à* la Beck).

In order to verify Beck's theorem you have to verify:

(i) U has a left adjoint (this is (1))
(ii) U has U-split coequalizers (this is (5))
(iii) U preserves coequalizers of U-split pairs.
(iv) U reflects coequalizers of U-split pairs.

For (iii) and (iv) go back to diagram (**??**): reflects means: take a pair $\circ \rightrightarrows \circ$ to $\mathcal{A}^{\mathbb{T}}$, then see down and use i.

So, U is a monad (*à* la Beck), and hence

$$\begin{tikzcd} \mathcal{V} \cong \mathcal{A}^{\mathbb{S}} \arrow[dr] & & \mathcal{A}^{\mathbb{T}} \arrow[ll, "\hat{i}"'] \\ & \mathcal{A} & \end{tikzcd}$$

$\hat{i}$ is induced by

a map of monads $\mathbb{T} \longrightarrow \mathbb{S}$
or a map of theories $\Theta_{\mathbb{T}} \longrightarrow \Theta_{\mathbb{S}}$
or $TA \longrightarrow SA$ being a split epi.

The reason is

$$U^{\mathbb{T}} F^{\mathbb{T}} A = TA \qquad U(\hat{i}(F^{\mathbb{T}}A)) = U(F(A))$$

Now using (82) with $V = FA$ we get

$$F^{\mathbb{T}}(A) \longrightarrow i(FA)$$

So

$$TA = U^{\mathbb{T}} F^{\mathbb{T}} A \longrightarrow U^{\mathbb{T}} i FA = UFA = U\hat{i}F^{\mathbb{T}}A$$

Canonical $\mathbb{T} \longrightarrow \mathbb{S}$ has each $TA \longrightarrow SA$ a split epi

THE CASE OF $\mathcal{S}ets$. The last thing is to go to the case $\mathcal{A} = \mathcal{S}ets$ and read off:

THEOREM 6 (Birkhoff). *When $\mathcal{A} = \mathcal{S}ets$, $\mathcal{V} \subseteq \mathcal{S}ets^{\mathbb{T}}$ is an abstract Birkhoff subcategory* if and only if *$\mathcal{V}$ is HSP-closed in $\mathcal{S}ets^{\mathbb{T}}$.*

PROOF. $\Rightarrow$) easy because $\mathcal{V}$ has all these creatures and i reflects them.

$\Leftarrow$) How given HSP, you achieve (B1) and (B2)? Condition (B1) is literally condition H. Strategy: Set (A, α) be any $\mathbb{T}$-algebra in $\mathcal{S}ets^{\mathbb{T}}$.

For each $V \in |\mathcal{V}|$ and each $f : (A,\alpha) \longrightarrow i(V)$, form

$$\equiv_{(V,f)} \subseteq ((A,\alpha))^2 = (A \times A, \lceil(\alpha,\alpha)\rceil)$$

and using HSP, $(A,\alpha)/\equiv_{(V,f)} \in |\mathcal{V}|$. So take

$$E = \bigcap_{V,f} \equiv_{(V,f)} \subseteq (A \times A, \lceil(\alpha,\alpha)\rceil)$$

Claim: $A/E \in |\mathcal{V}|$, and it does the trick (B4) is asking for. Indeed, select judiciously representatives ofr a full spectrum of $\equiv_{(V,f)}$ ... say (V_j, f_j), $j \in J$.

$$(A,\alpha) \xrightarrow{\langle \cdots f_j \cdots \rangle_{j\in J}} \prod_{j\in J} i(V_j)$$

and $\prod_{j\in J} i(V_j) \in \mathcal{V}$ according to hte P-part of HSP. The congruence relation for this is:

$$E \equiv_{\langle \cdots f_j \cdots \rangle_{j\in J}} (A,\alpha)$$

$$\begin{tikzcd}
(A,\alpha) \arrow[rr] \arrow[dr] & & \prod_{j\in J} i(Vj) \\
& (A,\alpha)/E \arrow[ur, no head] &
\end{tikzcd}$$

□

Summarizing,

- For finitary operations and Δ-algebras get Birkhoff's theorem
- For infinitary bounded operations and Δ-algebras get a version of Słomiński's theorem.
- Monads in $\mathcal{S}ets$ (compact Hausdorff spaces, compact abelian groups, ...)

LECTURE 24

[Counter]examples and corollaries (12/2)

COMPOSITE MONADICITY (TRIPLEABLENESS) THEOREMS. When two functors are composable? The setting is always this:

$$
\begin{tikzcd}
\mathcal{X} \arrow[d, shift left, "U_0"] \arrow[dd, bend left=70, "U = U_1 \circ U_0"] \\
\mathcal{K} \arrow[u, shift left, "F_0"] \arrow[d, shift left, "U_1"] \\
\mathcal{A} \arrow[u, shift left, "F_1"] \arrow[uu, bend left=70, "F = F_0 \circ F_1"]
\end{tikzcd}
$$

with $\mathcal{X}$ monadic over $\mathcal{K}$ via U_0, F_0, and so on. $\mathcal{K}$ monadic over $\mathcal{A}$ via U_1, F_1, etc.

Let us see what do we need:

1. It is not hard to see that F will be adjoint to U if F_0, F_1 are adjoint to U_0, U_1.

2. We also need $\mathcal{X}$ has coequalizers for U-split pairs

3. preserve and reflect coequalizers (?? this part needs to be completed)

In this setting, $\mathcal{X}$ *will* be monadic over $\mathcal{A}$ via U, F, etc. if alternatively

1. If the coequalizer in $\mathcal{K}$ for U_1-split pairs of $\mathcal{K}$-morphisms are in fact already split coequalizers in $\mathcal{K}$ (some people would say U_0 is *very tripeable*) or
2. If U_0 "preserves and reflects" coequalizers in $\mathcal{X}$ and if $\mathcal{X}$ has coequalizers for pairs U_0 of which has coequalizers (U_0 is *crudely tripeable*).

LECTURE 24

[Counter]examples and corollaries (12/2)

COMPOSITE MONADICITY (TRIPLEABLENESS) THEOREMS. When two functors are composable? The setting is always this:

$$
\begin{tikzcd}
\mathcal{X} \arrow[d, shift left, "U_0"] \arrow[dd, bend left=60, "U = U_1 \circ U_0"] \\
\mathcal{K} \arrow[u, shift left, "F_0"] \arrow[d, shift left, "U_1"] \\
\mathcal{A} \arrow[u, shift left, "F_1"] \arrow[uu, bend left=60, "F = F_0 \circ F_1"]
\end{tikzcd}
$$

with $\mathcal{X}$ monadic over $\mathcal{K}$ via U_0, F_0, and so on. $\mathcal{K}$ monadic over $\mathcal{A}$ via U_1, F_1, etc.

Let us see what do we need:

1. It is not hard to see that F will be adjoint to U if F_0, F_1 are adjoint to U_0, U_1.

2. We also need $\mathcal{X}$ has coequalizers for U-split pairs

3. preserve and reflect coequalizers (?? this part needs to be completed)

In this setting, $\mathcal{X}$ *will* be monadic over $\mathcal{A}$ via U, F, etc. if alternatively

1. If the coequalizer in $\mathcal{K}$ for U_1-split pairs of $\mathcal{K}$-morphisms are in fact already split coequalizers in $\mathcal{K}$ (some people would say U_0 is *very tripeable*) or
2. If U_0 "preserves and reflects" coequalizers in $\mathcal{X}$ and if $\mathcal{X}$ has coequalizers for pairs U_0 of which has coequalizers (U_0 is *crudely tripeable*).

In a schema:

$$T\Big\downarrow \qquad \begin{matrix} CT \\ \Leftarrow \\ T \end{matrix} \qquad \begin{matrix} \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \end{matrix} \qquad \begin{matrix} T \\ \Rightarrow \\ VTT \end{matrix} \qquad \Big\downarrow T \tag{83}$$

Why such things are necessary to tripeability?

COUNTEREXAMPLE 1. Let us consider the following diagram:

$$\begin{tikzcd}
\text{Torsion free Ab. Groups} \arrow[d, shift left, "U_{\text{torsion}}"] \\
\text{Ab. Groups} \arrow[u, shift left, "F_{\text{Torsion free quotient}}"] \arrow[d, shift left] \\
\mathcal{S}ets \arrow[u, shift left]
\end{tikzcd}$$

Here the triple in Abelian Groups is $T(G) = F(G)$, $TT(G) = T(G)$, where F is the torsion free quotient. So $\mu : TT \longrightarrow T$ is the identity, and $\eta : \mathrm{id} \longrightarrow T$ is $G \mapsto T(G)$. What does it mean here to have an algebra?

$$\begin{tikzcd}
 & T(G) \arrow[d, "\gamma"] \\
G \arrow[ur, "\eta_G"] & G \arrow[l, phantom, "="]
\end{tikzcd}$$

η_G must be 1-1 and onto. So an algebra for the resulting monad is nothing else that to be a torsion free group from the beginning. BUT,

$$\begin{tikzcd}
\text{Torsion free Ab. Gr.} \arrow[d, shift left, "U"] \\
\mathcal{S}ets \arrow[u, shift left, "F"]
\end{tikzcd} \tag{84}$$

is NOT monadic. What fails?

$$
\begin{tikzcd}
ED \arrow[r, shift left, "k"] \arrow[r, shift right, "n"'] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \\
\circ & \circ \arrow[l, "(l,[\frac{l}{2}]) \leftarrow l"'] & \circ \arrow[l, "0 \leftarrow 0,1 \leftarrow 1"']
\end{tikzcd}
\tag{85}
$$

where $ED = \{(k,n) | n + k \in \mathbb{Z}_2\}$ are the even differences.

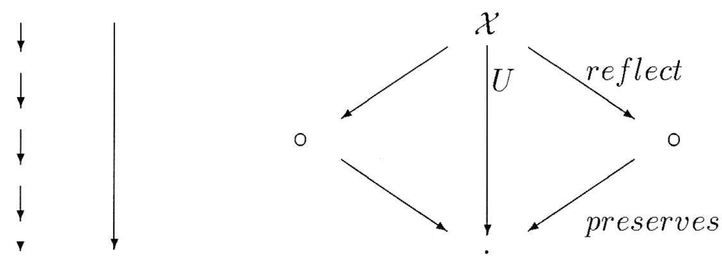
(85) is U-split in $\mathcal{S}ets$ of the original pair in (84). Now,

$$
\begin{tikzcd}
 & & A \\
ED \arrow[r, shift left] \arrow[r, shift right] & \mathbb{Z} \arrow[ur] \arrow[r] & \{0\} \arrow[u]
\end{tikzcd}
$$

If A is torsion free, then the $\{0\}$ group is the only so the functor U doesn't preserve the coequalizers for the U-split pairs ($\mathbb{Z}_2$ must have been the coequalizers in Abelian Groups). Condition (2) fails, condition (1) also fails.

This show basically that something must be added to the sole condition of both U_0 and U_1 been monadic.

Now coming back to diagram (83). If you combine VTT and CT in various ways:



One could ask bizarre situations. One is this: Suppose (just for the sake of easiness) that there are no empty $\mathbb{S}$-algebras and no empty $\mathbb{T}$-algebras. Denote by $\mathcal{S}ets_*$ the category of non-emtpy sets.

$$
\begin{tikzcd}
\mathcal{S}ets^{\mathbb{S}} \times \mathcal{S}ets^{\mathbb{T}} \arrow[r] \arrow[d, "U^{\mathbb{S}} | U^{\mathbb{T}}"'] \arrow[dr, ".monadic"] \arrow[dd, bend right=60, ".monadic"'] & \mathcal{S}ets_* \times \mathcal{S}ets^{\mathbb{T}} \arrow[d] \\
\mathcal{S}ets \times \mathcal{S}ets \arrow[d, "no"] & \mathcal{S}ets_* \times \mathcal{S}ets_* \arrow[l, "no"'] \arrow[d, ".monadic"] \\
\mathcal{S}ets & \mathcal{S}ets_* \arrow[l]
\end{tikzcd}
$$

Is it monadic? Yes, but the proof uses (B1) and (B2) in some places.

COUNTEREXAMPLE 2. Let Banach the real line with continuous real linear transformations of norm less than 1.

$$\begin{tikzcd} \mathrm{Banach} \arrow[d, shift left, "D = \mathrm{Ban}(\mathbb{R}, -)"] \\ \mathcal{S}ets \arrow[u, shift left, "l_1"] \end{tikzcd}$$

where $D = \mathrm{Ban}(\mathbb{R}, -) =$ unit disc of (-).

$$\mathrm{Ban}(l_1(n), V) \cong \mathcal{S}ets(n, D(V))$$

We have that the image under D of all continuous linear transformations from $l_1(n)$ to V is isomorphic to $D(V^n)$. Could this be tripeable? NO. The triple

$$n \xrightarrow[i \mapsto (e_i)_j = \delta_{ij}]{} D(l_1(n)) \xleftarrow[\mu]{} D(l_1(D(l_1(n))))$$

The notations become cumbersome. Call $n(i) = [i]$, so the elements of $D(l_1(n))$ are of the form

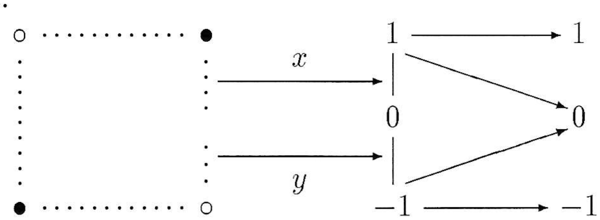
$$\sum_{j \in n} a_j[j]$$

Hence the elements of $D(l_1(D(l_1(n))))$ are:

$$\sum_{j \in Dl_1(n)} b_j \left[\sum_{i \in n} a_i[j] \right]$$

BUT, this is not monadic:

Take the unit disc. Take the open unit disc. It is a subalgebra of the unit disc.



This is a split coequalizer in Ban. What fails? Congruence relation bussiness.

LECTURE 25

Rank (revisited) (12/4)

In the context of past lectures, fix $\mathcal{A} = \mathcal{S}ets$ and focus on triples, monads, theories, categories with adjoint pairs, ... over $\mathcal{S}ets$. Let $\mathcal{S}ets \xrightarrow{\theta} \Theta$, and define:

$$\begin{aligned}
\mathrm{Th}_{\mathfrak{r}}(\Theta, \theta) :& \\
& \forall n \in \mathcal{S}ets \;\; \forall \omega \in \Theta(1, n) \\
& \exists k \in \mathcal{S}ets : \; ||k|| < \mathfrak{r} \\
& \exists f \in \mathcal{S}ets(k, n) \\
& \exists \lambda \in \Theta(1, k)
\end{aligned}$$

such that

(86)
$$\begin{tikzcd}
1 \arrow[rr, "\omega"] \arrow[dr, "\lambda"'] & & \omega \\
& k \arrow[ur, "\theta(f)"'] &
\end{tikzcd}$$

(If u is an n-ary operation, this is saying that only k variables matters, the others remain unchanged). This is the rank of a theory.

$$(\text{free algebra})_{\mathfrak{r}}(\mathcal{X} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{S}ets) : \forall n U(F(n)) = \forall n \bigcup_{\substack{||k||<\mathfrak{r} \\ f:k\longrightarrow n}} U(F(f))(UF(k))$$

For a triple $\mathbb{T} = (T, \eta, \mu)$,

$$\text{triple}_{\mathfrak{r}}(\mathbb{T}) \text{ iff } \forall n T(n) = \forall n \bigcup_{\substack{||k||<\mathfrak{r} \\ f:k\longrightarrow n}} T(f)(T(k))$$

When a theory and a triple goes hand by hand, i.e. when

$$\Theta = \Theta_{\mathbb{T}}$$
$$\Theta(1, n) \cong \mathcal{S}ets(1, T(n))$$

both definitions coincide.

?. Given a theory $\theta : \mathcal{S}ets \longrightarrow \Theta$, mean by its $\mathfrak{r}$-truncation the full image $\Theta_\mathfrak{r}$ of the restriction

$$
\begin{tikzcd}
\mathcal{S}ets_\mathfrak{r} \arrow[r] \arrow[dr, "\theta_\mathfrak{r}", "\mathfrak{r}\text{-truncation}"'] & \mathcal{S}ets \arrow[r, "\theta"] & \Theta \\
 & \Theta|_\mathfrak{r} \arrow[ur] &
\end{tikzcd}
$$

where $\mathcal{S}ets_\mathfrak{r}$ means sets of cardinality less than $\mathfrak{r}$. When one speaks of algebras, i.e. a functor $\Theta \longrightarrow \mathcal{S}ets$, we obtain a composition

$$
\begin{tikzcd}
\Theta_\mathfrak{r}\text{-}\mathcal{A}lg \arrow[rr] \arrow[d] & & \mathcal{S}ets^{\Theta_\mathfrak{r}^{op}} \arrow[d, "\text{comp. w/ } \theta_\mathfrak{r}"] \\
\mathcal{S}ets \arrow[r, "Y"'] & \mathcal{S}ets^{\mathcal{S}ets^{op}} \arrow[r, "\text{restr. to } \mathfrak{r}"'] & \mathcal{S}ets_\mathfrak{r}^{op}
\end{tikzcd}
$$

THEOREM 7. *If $(\mathcal{X}, FU)$ gives rise to a monad $\mathbb{T}$ and theory $\Theta_\mathbb{T}$ which in turn yield $\mathcal{S}ets^\mathbb{T}$, then the following are equivalent:*

1. *rank $\mathfrak{r}$ for the objects of $\mathcal{X}$*
2. *rank $\mathfrak{r}$ for the associated monad ($\mathbb{T}$ has rank $< \mathfrak{r}$)*
3. *rank $\mathfrak{r}$ for the associated theory ($\Theta_\mathbb{T}$ has rank $< \mathfrak{r}$)*
4. *rank $< \mathfrak{r}$ for $U^\mathbb{T}F^\mathbb{T}$ in $\mathcal{S}ets^\mathbb{T}$*
5. *(background for this):*

$$
\begin{tikzcd}
\theta\text{-}\mathcal{A}lg \arrow[r] \arrow[dr, dotted] \arrow[ddr] & \mathcal{S}ets^{\Theta^{op}} \arrow[drr] & & \\
 & \Theta_\mathfrak{r}\text{-}\mathcal{A}lg \arrow[rr] \arrow[d] & & \mathcal{S}ets^{\Theta_\mathfrak{r}^{op}} \arrow[d, "\textit{comp. w/ } \theta_\mathfrak{r}"] \\
 & \mathcal{S}ets \arrow[r, "Y"'] & \mathcal{S}ets^{\mathcal{S}ets^{op}} \arrow[r, "\textit{restr.}"'] & \mathcal{S}ets_\mathfrak{r}^{op}
\end{tikzcd}
$$

So there is always a way to compare θ-$\mathcal{A}lg$ and $\Theta_\mathfrak{r}$-$\mathcal{A}lg$. (5) is saying that the functor that lose sight of th high powers is nonetheless an isomorphism.

6. $\Theta_{\mathfrak{r}}\text{-}\mathcal{A}lg \longrightarrow \Theta_{\mathbb{T}}|_{\mathfrak{r}}\text{-}\mathcal{A}lg$ *is an isomorphism.*

where $\mathfrak{r}$ *is a regular cardinal (see definition in lect ??).*

PROOF. (1) $\Rightarrow$ (2) $\cdots \Rightarrow$ (5) $\Rightarrow$ (1)

(1) $\Rightarrow$ (2). Just note $T = UF$ when applied to f, n, k.

(2) $\Rightarrow$ (3). Since (see (86)) every n-ary operation is at most n-ray: is k-ary for some k.

(3) $\Rightarrow$ (4).

$$\Theta(1, n) \cong \mathcal{S}ets(1, Tn) = \mathcal{S}ets(1, U^{\mathbb{T}}F^{\mathbb{T}}n) = \mathcal{S}ets^{\mathbb{T}}(1, U^{\mathbb{T}}F^{\mathbb{T}}n)$$

(4) $\Rightarrow$ (5). Why the dotted arrow in the diagram (in (5) above) is an iso? Given a $\Theta_{\mathfrak{r}}$-algebra (A, a) from which one has meaning for $\mathrm{a} * \lambda \in A$, for $\mathrm{a} \in A^k$ and $\lambda \in \Theta(1, k) = T(k)$. So long as $||k|| < \mathfrak{r}$ wish now to have meaningful and well behaved expressions $\mathrm{a} * \omega$, for $\mathrm{a} \in A^n$ and $\omega \in \Theta_{\mathbb{T}}(1, n) = T(n)$. How to do it?

$$\omega \in T(n) = \bigcup_{\substack{||k|| < \mathfrak{r} \\ f : k \longrightarrow n}} T(f)(T(k))$$

There is $k, \lambda \in T(k), k \xrightarrow{f} n$, such that

$$\omega = T(f)(\lambda) = \theta(f) \circ \lambda$$

(from the perspective of the theory). So define $\mathrm{a} * \omega \in A$ by

$$\mathrm{a} * \omega = \mathrm{a} * (\theta(f) \circ \lambda) = (a * \theta(f)) * \lambda = (a \circ f) * \lambda$$

This makes sense:

$$\begin{tikzcd} & n \arrow[dr, "\mathrm{a}"] & \\ k \arrow[ur, "f"] \arrow[rr, "\mathrm{a} \circ f"'] & & A \end{tikzcd}$$

There are tedious verifications to check. (This is the only way of having big n-ary operations in Θ-algebras with underlying sets of small size $\mathcal{S}ets_{\mathfrak{r}}$).

(5) $\Rightarrow$ (1). A more general fact:

$$\begin{tikzcd} T\text{-}\mathcal{A}lg \arrow[rr] \arrow[d, shift left, "U"] & & \mathcal{S}ets^{T^{op}} \arrow[d] \\ \mathcal{S}ets \arrow[u, shift left, "F"] \arrow[r] & \mathcal{S}ets^{\mathcal{S}ets^{op}} \arrow[r] & \mathcal{S}ets^{\mathcal{S}ets_{\mathfrak{r}}^{op}} \end{tikzcd}$$

If $k < \mathfrak{r}$, then $F(k) = T(1,k) = \mathrm{nat}(U^k, U)$. For general k, need to see that the class $\mathrm{nat}(U^k, U)$ is a set, namely

$$\bigcup_{\substack{\|k\|<\mathfrak{r} \\ f:k\longrightarrow n}} U^f(\mathrm{nat}(U^k, U))$$

where U^f is defined as usual: $(n \to UA) \overset{-\circ f}{\rightsquigarrow} (k \to UA)$ in

$$U^n \xrightarrow{U^f} U^k \xrightarrow{\lambda} U$$

The set $\mathrm{nat}(U^k, U)$ plays here the rol of free algebra for $\mathrm{nat}(U^n, U)$. If you mimic the proof of the Θ.. truncation,...

$\square$

LECTURE 26

$\mathcal{V}$- Categories (12/9)

References for this lecture:

1. Proceedings Conference on Categorical Algebra, La Jolla, 1965, Springer-Verlag, pp. 421 - end.
2. Springer LNM 99, pp. 384-ff
3. Springer LNM 195, pp. 209-ff

QUESTION. What must a category $\mathcal{V}$ be doing (be equiped with) in order to let one use the objects of $\mathcal{V}$ as potential hom-objects for other categories?

Some examples of the kind of creatures I may have in mind for $\mathcal{V}$:

- $\mathcal{V} = \mathcal{S}ets$ (ordinary categories)
- $\mathcal{V} = \mathbf{2} = \{0, 1\}$ (posets)
- $\mathcal{V} = \mathbb{R}^+ \cup \{+\infty\} = [0, \infty]$ with $\geq$ (not necessarily separated symmetric metric spaces)
- $\mathcal{V} = \mathcal{A}b\mathcal{G}p$
- $\mathcal{V} = \mathcal{B}an$ (normed bounded (≤ 1) transformations)

Answering the question above: we should have

$$\langle \mathcal{A}(B, C), \mathcal{A}(A, B)\rangle \longrightarrow \mathcal{A}(A, C) \tag{87}$$

$$\langle\rangle \longrightarrow \mathcal{A}(A, A) \tag{88}$$

This is all we need to ask. So we need to know what is the function of two variables and the function of $\langle\rangle$ variable in the realm where this objects live.

Categories $\mathcal{V}$ should come equiped with a (well behaved) explanation (definition) (understanding) of what is to be meant by a "morphism of n variables" ($n \in \mathbb{N}$)

$$\langle A_1, \ldots, A_n\rangle \longrightarrow B$$

where $A_i \in |\mathcal{V}|$, $1 \leq i \leq n$ (when $i = 0$, I mean literally the empty string).

EXAMPLES. In $\mathcal{S}ets$, for (87) just take the cartesian product

$$\langle A_1, A_2\rangle \longrightarrow B \quad \text{is} \quad A_1 \times A_2 \longrightarrow B$$

and for the identity (88) take

$$\langle\rangle \longrightarrow B \quad \text{is} \quad b \in B$$

In $\mathcal{A}b\,\mathcal{G}p$,

$$\langle A_1, A_2\rangle \longrightarrow B \quad \text{is} \quad A_1 \otimes A_2 \longrightarrow B$$

and

$$\langle\rangle \longrightarrow B \quad \text{is} \quad \mathbb{Z} \longrightarrow B$$

TENTATIVE DEFINITON OF MONOIDAL CATEGORY. When one has a category $\mathcal{V}$, let us write

$$\mathcal{M}(\mathcal{V}) = \text{free monoid generated by the objects of } \mathcal{V}$$

and by

$$(\langle A_1, \ldots, A_n\rangle, B)$$

the sets of maps from $A \in |\mathcal{M}(\mathcal{V})|$ to objects $B \in \mathcal{V}$, such that

$$(\langle A\rangle, B) = \mathcal{V}(A, B)$$

and if we have

$$\begin{aligned} \langle A_1, \ldots, A_{n_1}\rangle &\xrightarrow{f_1} B_1 \\ \langle A_{n_1+1}, \ldots, A_{n_1+n_2}\rangle &\xrightarrow{f_2} B_2 \\ &\vdots \\ \langle A_{n_1+n_2+\cdots+n_{k-1}+1}, \ldots, A_{n_1+n_2+\cdots+n_k}\rangle &\xrightarrow{f_k} B_k \end{aligned}$$

and

$$\langle B_1, \ldots, B_k\rangle \longrightarrow C$$

one presumably wants a composition

$$g(f_1, \ldots, f_k) : \langle A_1, \ldots, A_{n_1+\cdots+n_k}\rangle \longrightarrow C \tag{89}$$

Asking to $\mathcal{M}(\mathcal{V})$ to be a category is asking a little more than this and at the same time a little less (i.e. less technicality).

PROPOSE. Ask $\mathcal{M}(\mathcal{V})$ to be a category and to be monoidal in the strict sense using the monoidal structure of $|\mathcal{M}(\mathcal{V})|$.

In order to formulate this concept in the right way we need to introduce some notions.

DEFINITION 13. Say that a category *has a multiplication (is a monoidal category) in the strictest sense* if there are

1. A functor[1] $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$

[1]The $\otimes$ notation is to follow the convention

2. An object $I \in \mathcal{M}$ such that the following two diagrams commute

$$\begin{tikzcd}
\mathcal{M}\times\mathcal{M}\times\mathcal{M} \arrow[r, "\mathcal{M}\times\otimes"] \arrow[d, "\otimes\times\mathcal{M}"'] & \mathcal{M}\times\mathcal{M} \arrow[d, "\otimes"] \\
\mathcal{M}\times\mathcal{M} \arrow[r, "\otimes"'] & \mathcal{M}
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{M}\times\mathcal{M} \arrow[dr, "\otimes"] & \\
\mathcal{M} \arrow[ur, "I\times\mathrm{id}"] \arrow[rr, "\mathrm{id}"] \arrow[dr, "\mathrm{id}\times I"'] & & \mathcal{M} \\
 & \mathcal{M}\times\mathcal{M} \arrow[ur, "\otimes"'] &
\end{tikzcd}$$

THE MONOIDAL Δ. Let Δ be the category of finite Von Neumann ordinals (the old familiar $1, 2, 3, \ldots$) with order-preserving functions as the Δ-morphisms, and with the followign $+$ as $\otimes$: if $\alpha : n \longrightarrow N$ and $\beta : k \longrightarrow K$, then define

$$(\alpha\otimes\beta)(i) = \begin{cases} \alpha(i) & \text{if } i \le n \\ \beta(i-n)+N & \text{if } i > n \end{cases}$$

It is easy to see that $\otimes$ so defined is associative.

Why we need this? We want to deal in a clean way with the subscripts that arise in the formulation of equation (89). We want a morphism

$$|\mathcal{M}(\mathcal{V})| \longrightarrow |\Delta|$$

to be the effect on objects of a full fledged functor $\mathcal{M}(\mathcal{V}) \longrightarrow \Delta$ that *preserves* $\otimes$-products, and satisfies in addition:

$$\forall n \in |\Delta|,\ \forall 1 \le k \le |\Delta|,\ \forall\,\mathrm{length}(A) = n,\ \forall\,\mathrm{length}(B) = k,$$
$$\forall f \in \mathcal{M}(\mathcal{V}),\ f : \langle A_0, \ldots, A_{n-1}\rangle \longrightarrow \langle B_0, \ldots, B_{k-1}\rangle$$
$$\exists!\ \text{maps}\ \ f_j : \langle \cdots A_i \cdots \rangle_{i\in\alpha^{-1}(j)} \longrightarrow B_j\ \ \text{such that}\ \ f = f_1 \otimes \cdots \otimes f_{k-1}$$

This is the cleanest way to express equation (89)

The easiest example of such a category comes from one that was already a monoidal category, but there are other illustrations as well.

Such a $\mathcal{V}$ (already with $\mathcal{M}(\mathcal{V})$ information) is a "multilinear category". Multilinear categories are the most able recipients of (hom?) homomorphisms.

$\mathcal{V}$-CATEGORIES.[2] If $\mathcal{V}$ *is* a multilinear category, then by a $\mathcal{V}$-category $\mathcal{X}$ is meant a class $\mathcal{X}$ of objects along with a double indexed class $\mathcal{X}(A,B) \in \mathcal{V}$ of "hom-objects" $(A, B \in |\mathcal{X}|)$ whose composition rule is

$$\langle \mathcal{X}(A,B), \mathcal{X}(A,B)\rangle \longrightarrow \mathcal{X}(A,C)$$

and unit maps

$$\langle\rangle \xrightarrow{e_A} \mathcal{X}(A,A)$$

as provided by $\mathcal{M}(\mathcal{V})$ such that composition is associative and units behave like identity maps.

If you are lucky $\mathcal{V}$ itself will be a $\mathcal{V}$-category. Yo do not want it to be randomly, but in such a way

$$\mathcal{M}(\mathcal{V})(A, \underline{\mathcal{V}}(C,D)) = \mathcal{M}(\mathcal{V})(\langle A, C\rangle, D)$$

(note that we are seen in the equation above $\underline{\mathcal{V}}(C,D)$ as an $\mathcal{V}$-object). Even better

$$\mathcal{M}(\mathcal{V})(\langle A_1, \dots, A_n\rangle, \underline{\mathcal{V}}(C,D)) = \mathcal{M}(\mathcal{V})(\langle A_1, \dots A_n\rangle \otimes C, D)$$

Now, $\mathcal{V}$ itself is a $\mathcal{V}$-category in a completely compatible way.

CLOSED CATEGORIES. When $\mathcal{V}$ is a $\mathcal{V}$-category in this way one says $\mathcal{V}$ is a *closed category.*

Another way a category could be multilinear (without being a strict monoidal category), is maybe

$$\mathcal{M}(\mathcal{V})(\langle A_1, \dots, A_n\rangle, -)$$

is representable functor as a functor

$$\mathcal{V} \longrightarrow \mathcal{S}ets$$

with $A_1 \otimes A_2 \in |\mathcal{V}|$. It is very difficult to assure

$$(A_1 \otimes A_2) \otimes A_3 = A_1 \otimes (A_2 \otimes A_3)$$

They will be isomorphic, but there is no guarantee that they be equal. This gives rise to *monoidal categories* that are on the references I gave at the beginning:

1. They were defined in the article in Kelly, MacLane.
2. The second reference deals with ....
3. Why the proofs in (1) and (2) were so similar.

[2]Here we find again the story we already saw about natural equivalences: you need the concept of natural equivalence, so you need the notion of natural transformation, but in order to state it you need the notion of functor and category. Here is the same: in order to define $\mathcal{V}$-categories, we need to go through many other concepts before.

So we ended as we began in the first lecture, and this stage is finished. Next, we would have to walk the same way again, but at a higher level.